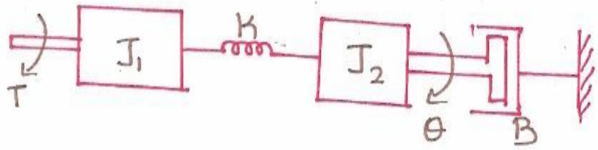
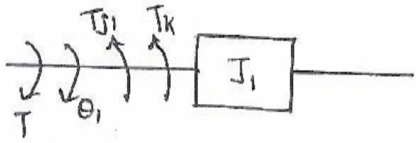
PART-B.

1) Write the differential equation for a mechanical system shown in figure. Also obtain the transfer function $\frac{\Theta(s)}{T(s)}$. [M|J - 2014]



Solution:-

Free body diagram for J_1,



$$T_{J1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_K = K(\theta_1 - \theta)$$

By Newton's second law,

$$T_{J1} + T_K = T$$

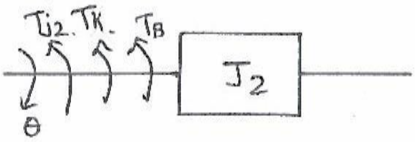
$$J_1 \frac{d^2\theta}{dt^2} + K(\theta_1 - \theta) = T$$

Taking laplace transform,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - \theta(s)K = T(s)$$

$$\theta_1(s)[J_1 s^2 + K] - K\theta(s) = T(s) \rightarrow (1)$$

Free body diagram for J_2,



$$T_{j2} = J_2 \frac{d^2\theta}{dt^2}$$

$$T_K = K[\theta - \theta_1]$$

$$T_B = B\frac{d\theta}{dt}$$

By Newton's second law,

$$T_{j2} + T_K + T_B = 0$$

$$J_2\frac{d^2\theta}{dt^2} + K(\theta - \theta_1) + B\frac{d\theta}{dt} = 0$$

Taking laplace transform,

$$J_2 s^2\theta(s) + K\,\theta(s) - K\,\theta_1(s) + Bs\,\theta(s) = 0$$

$$\theta(s)[J_2 s^2 + K + Bs] = K\theta_1(s).$$

$$\theta_1(s) = \frac{\theta(s)[J_2 s^2 + K + Bs]}{K} \rightarrow ②$$

Substitute eqn ② in ①,

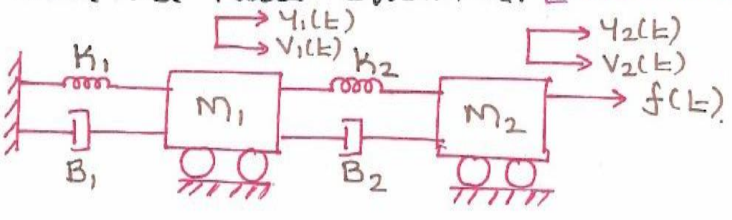
$$\theta(s)\frac{[J_2 s^2 + K + Bs]}{K}[J_1 s^2 + K] - K\theta(s) = T(s)$$

$$\theta(s)\left[\frac{(J_2 s^2 + K + Bs)(J_1 s^2 + K) - K^2}{K}\right] = T(s)$$

Transfer function,

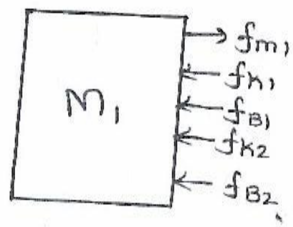
$$\boxed{\frac{\theta(s)}{T(s)} = \frac{K}{[J_2 s^2 + K + Bs][J_1 s^2 + K] - K^2}}$$

2) Consider a mechanical system shown in figure. Write the differential equation describing the dynamic system and draw the electrical analogous circuit for the system. Verify them by writing mesh and node equations. [M|J-2014]



Solution:-

Free body diagram for mass M_1,



$$f_{m1} = M_1 \frac{d^2 y_1(t)}{dt^2}$$

$$f_{k1} = K_1 y_1(t)$$

$$f_{b1} = B_1 \frac{dy_1(t)}{dt}$$

$$f_{k2} = K_2 [y_1(t) - y_2(t)]$$

$$f_{b2} = B_2 \frac{d}{dt} [y_1(t) - y_2(t)]$$

By Newton's second law,

$$f_{m1} + f_{k1} + f_{b1} + f_{k2} + f_{b2} = 0$$

$$M_1 \frac{d^2 y_1(t)}{dt^2} + K_1 y_1(t) + B_1 \frac{dy_1(t)}{dt} + K_2 [y_1(t) - y_2(t)] + B_2 \frac{d}{dt} [y_1(t) - y_2(t)] = 0$$

Here, $\frac{d^2 x}{dt^2} = \frac{dv}{dt}$, $\frac{dx}{dt} = v$, $x = \int v\,dt$.

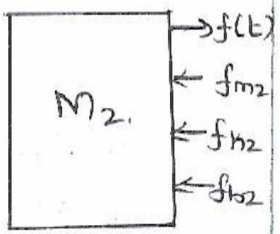
$$M_1 \frac{dv_1}{dt} + K_1 \int v_1 \, dt + B_1 V_1 + K_2 \int (V_1 - V_2) \, dt + B_2 (V_1 - V_2) = 0 \rightarrow (1)$$

Free-body diagram for mass M_2,

$$f_{m2} = M_2 \frac{d^2 y_2(t)}{dt^2}$$

$$f_{k2} = K_2 [y_2(t) - y_1(t)]$$

$$f_{b2} = B_2 \frac{d}{dt} [y_2(t) - y_1(t)]$$



By Newton's second law,

$$f_{m2} + f_{k2} + f_{b2} = f(t)$$

$$M_2 \frac{d^2 y_2(t)}{dt^2} + K_2 [y_2(t) - y_1(t)] + B_2 \frac{d}{dt} [y_2(t) - y_1(t)] = f(t).$$

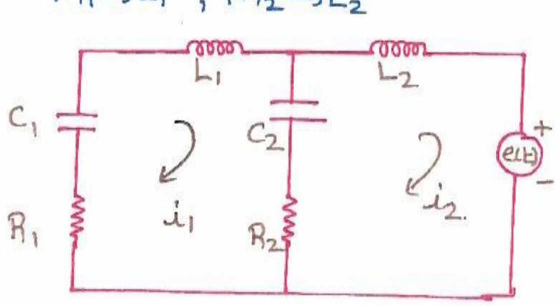
$$M_2 \frac{dv_2}{dt} = K_2 \int (V_2 - V_1) \, dt + B_2 (V_2 - V_1) = F(t) \rightarrow (2)$$

Force-Voltage analogous circuit:-

$f(t) \rightarrow e(t)$, $K_1 \rightarrow 1/C_1$, $K_2 \rightarrow 1/C_2$.

$V_1 \rightarrow i_1$, $V_2 \rightarrow i_2$, $B_1 \rightarrow R_1$, $B_2 \rightarrow R_2$

$M_1 \rightarrow L_1$, $M_2 \rightarrow L_2$



Mesh equation is given by,

$$L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + R_1 i_1 + \frac{1}{C_2}\int (i_1 - i_2)\, dt + R_2(i_1 - i_2) = 0 \rightarrow (3)$$

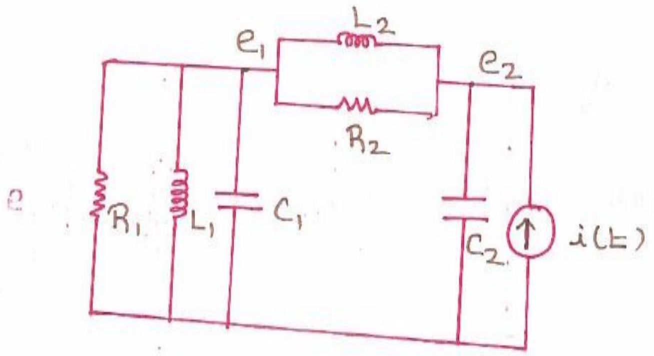
$$L_2 \frac{di_2}{dt} + \frac{1}{C_2}\int (i_2 - i_1)\, dt + R_2(i_2 - i_1) = e(t) \rightarrow (4)$$

Force - Current analogous circuit:-

$f(t) \rightarrow i(t)$ $\qquad K_1 \rightarrow 1/L_1$, $K_2 \rightarrow 1/L_2$

$V_1 \rightarrow e_1$, $V_2 \rightarrow e_2$ $\qquad B_1 \rightarrow R_1$, $B_2 \rightarrow R_2$

$M_1 \rightarrow C_1$, $M_2 \rightarrow C_2$



Node equation is given by,

$$C_1 \frac{de_1}{dt} + \frac{1}{L_1}\int e_1 dt + \frac{1}{R_1} e_1 + \frac{1}{L_2}\int (e_1 - e_2)\, dt + \frac{1}{R_2}(e_1 - e_2) = 0 \rightarrow (5)$$

$$C_2 \frac{de_2}{dt} + \frac{1}{L_2}\int (e_2 - e_1)\, dt + \frac{1}{R_2}(e_2 - e_1) = i(t) \rightarrow (6)$$

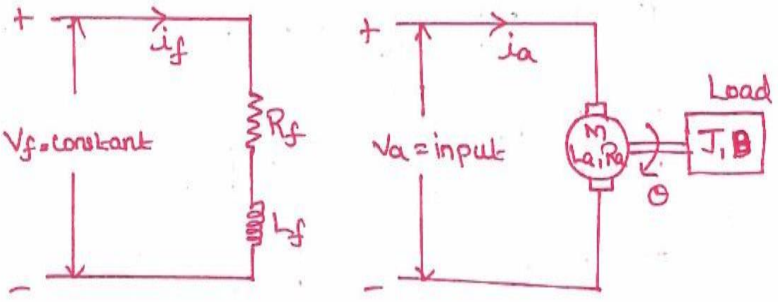
It is observed that the mesh equations (3) & (4) are similar to the differential equations (1) & (2) and also node equations (5) & (6) are similar to (1) & (2).

3) Derive the transfer function of armature control and field control of DC motor.

[N/D-2013, Apr/May-2016]

Transfer Function of Armature Controlled DC motor:-

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux in field winding. In armature controlled DC motor, the desired speed is obtained by varying the armature voltage.



By Kirchoff's voltage law,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \quad \rightarrow (1)$$

where,

$i_a \rightarrow$ armature current

$R_a \rightarrow$ armature resistance

$L_a \rightarrow$ armature inductance

$e_b \rightarrow$ back emf.

$V_a \rightarrow$ armature voltage.

Torque of DC motor is proportional to the product of flux and current. Since flux is constant, the torque is proportional to ia alone.

$$T \propto i_a$$

$$T = K_t i_a \rightarrow (2).$$

Where,

$T \rightarrow$ torque

$K_t \rightarrow$ torque constant.

The differential equation governing the mechanical system of motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \rightarrow (3)$$

Where,

$J \rightarrow$ moment of inertia.

$B \rightarrow$ Frictional co-efficient.

$\theta \rightarrow$ angular displacement.

The back emf of DC machine is proportional to speed.

$$e_b \propto \frac{d\theta}{dt}.$$

$$e_b = K_b\frac{d\theta}{dt} \rightarrow (4)$$

Where,

$K_b \rightarrow$ back emf constant.

Taking laplace transform for eqns ①, ②, ③ & ④,

$$I_a(s)R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad \rightarrow ⑤$$

$$T(s) = K_t I_a(s). \quad \rightarrow ⑥$$

$$Js^2\theta(s) + Bs\,\theta(s) = T(s) \quad \rightarrow ⑦$$

$$E_b(s) = K_b s\,\theta(s) \quad \rightarrow ⑧$$

Equating eqn ⑥ & ⑦,

$$K_t I_a(s) = Js^2\theta(s) + Bs\,\theta(s)$$

$$I_a(s) = \frac{\theta(s)[Js^2 + Bs]}{K_t} \quad \rightarrow ⑨$$

$$⑤ \Rightarrow \quad I_a(s)[R_a + L_a s] + E_b(s) = V_a(s)$$

$$I_a(s)[R_a + L_a s] + K_b s\theta(s) = V_a(s)$$

$$\frac{\theta(s)[Js^2 + Bs]}{K_t}[R_a + L_a s] + K_b s\,\theta(s) = V_a(s)$$

Transfer function,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(Js^2 + Bs)(R_a + L_a s) + K_b K_t s}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{R_a\left[1 + \frac{sL_a}{R_a}\right] Bs\left[1 + \frac{Js^2}{Bs}\right] + K_b K_t s}$$

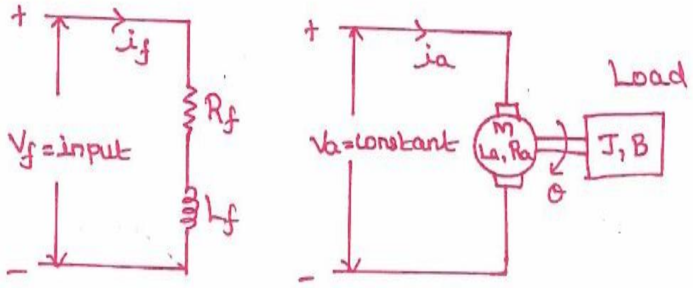
$$\boxed{\frac{\theta(s)}{V_a(s)} = \frac{K_t / R_aB}{s\left[(1+sT_a)(1+sT_m)+\frac{K_bK_t}{R_aB}\right]}}$$

where, $\frac{L_a}{R_a} = T_a$ = Electrical time constant.

$\frac{J}{B} = T_m$ = Mechanical time constant.

Transfer Function of Field controlled DC Motor:-

The speed of a DC motor is directly proportional to armature voltage and inversely proportional to flux. In field controlled DC motor, the armature voltage is kept constant and the speed is varied by varying the flux of the machine.



By Kirchoff's voltage law,

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad \rightarrow (1)$$

where,

$i_f \rightarrow$ Field current

$L_f \rightarrow$ Field inductance

$R_f \rightarrow$ field resistant

$V_f \rightarrow$ field voltage

The torque is proportional to product of flux and armature current. Since I_a is constant, the torque is proportional to flux alone.

$$T \propto i_f$$

$$T = K_f i_f \quad \rightarrow (2)$$

where, $K_f \rightarrow$ flux constant.

The differential equation governing the mechanical system of motor is,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad \rightarrow (3)$$

Taking laplace transform for eqns (1), (2) & (3),

$$R_f I_f(s) + L_f s I_f(s) = V_f(s) \quad \rightarrow (4)$$

$$T(s) = K_f I_f(s) \quad \rightarrow (5)$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \quad \rightarrow (6)$$

Equating eqns (5) & (6),

$$K_f I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = \frac{\theta(s)(Js^2+Bs)}{K_f} \rightarrow ⑦$$

$$④ \Rightarrow I_f(s)[R_f + sL_f] = V_f(s) \rightarrow ⑧$$

Substitute eqn ⑦ in ⑧,

$$\frac{s\,\theta(s)\,(Js+B)}{K_f}\,[R_f + sL_f] = V_f(s)$$

Transfer Function,

$$\frac{\theta(s)}{V_f(s)} = \frac{K_f}{s(R_f+sL_f)(Js+B)}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_f}{sR_f\left[1+\frac{sL_f}{R_f}\right]B\left[1+\frac{Js}{B}\right]}$$

$$\boxed{\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(1+sT_f)(1+sT_m)}}$$

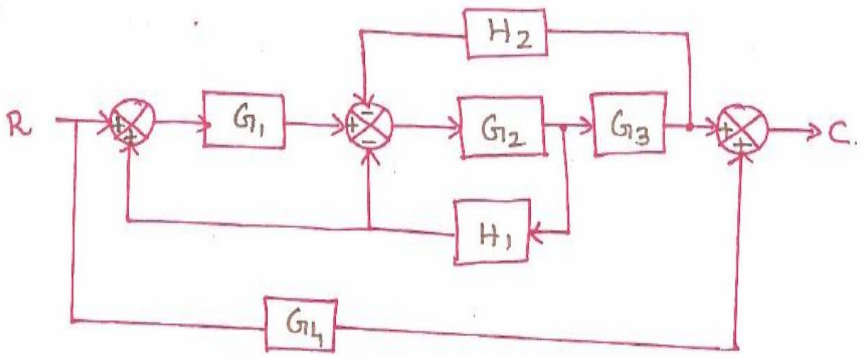
where,

$K_m = \dfrac{K_f}{R_f B}$ = Motor gain constant.

$T_f = \dfrac{L_f}{R_f}$ = Field time constant

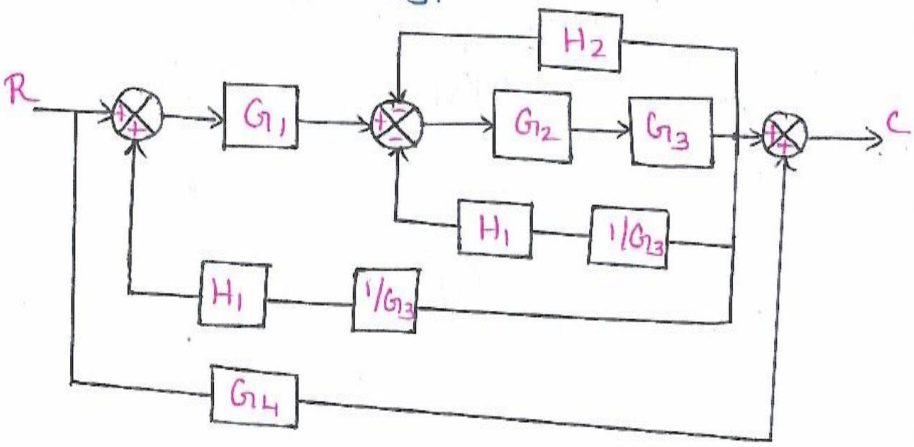
$T_m = \dfrac{J}{B}$ = Mechanical time constant.

4) Determine the transfer function by using block diagram reduction technique. [May-2013, May-2014, N/D-12]

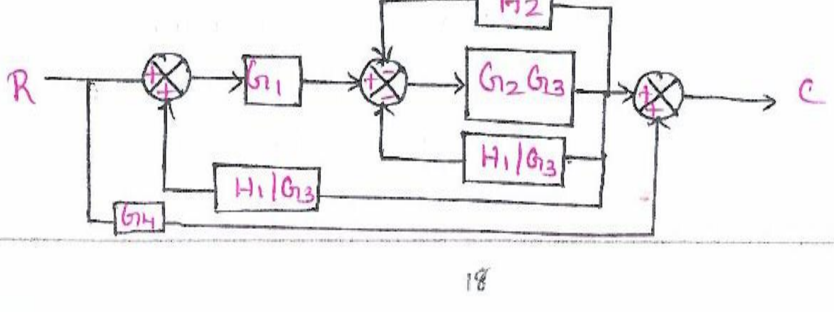


Solution:-

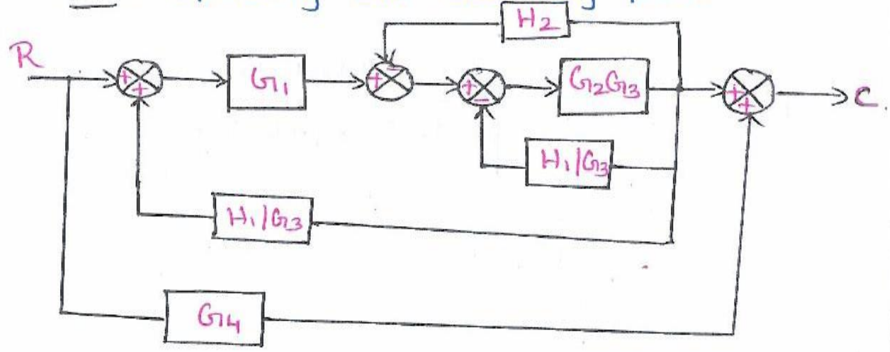
Step-1:- Moving the branch point after the block G_3.



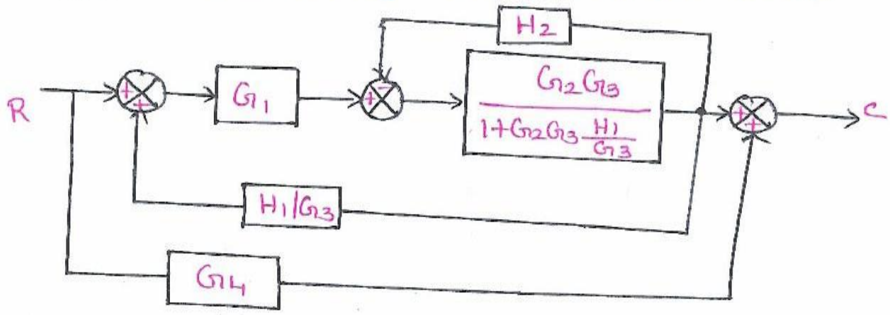
Step-2:- Combining the cascade block.



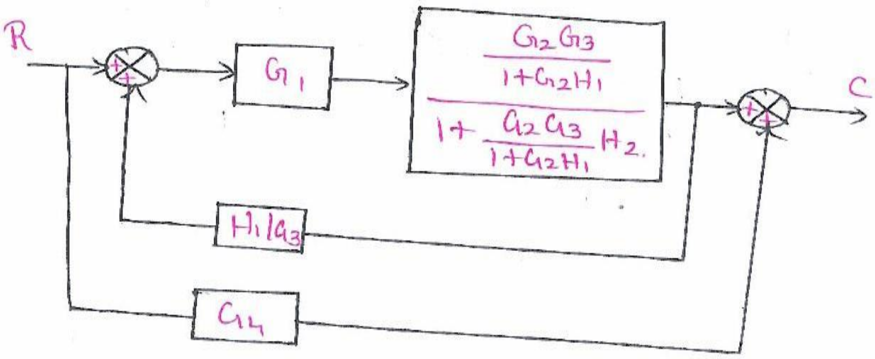
Step-3:- Splitting the summing point.



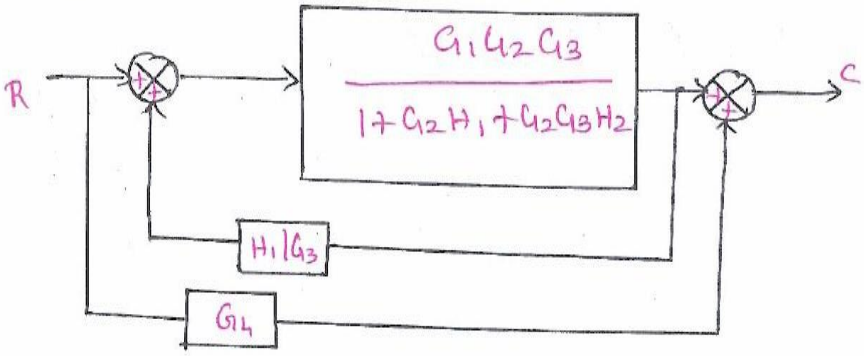
Step-4:- Eliminating the feedback path.



Step-5:- Eliminating the feedback path.



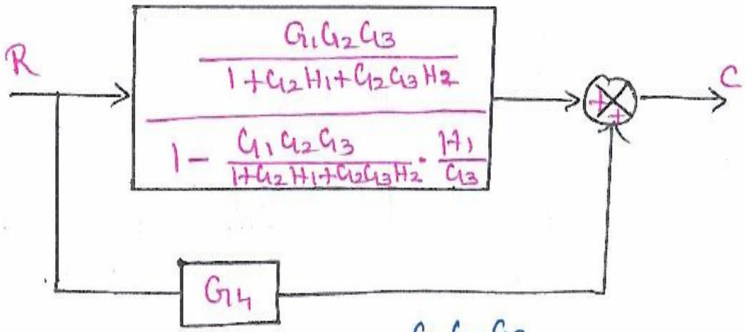
Step-6:- Combining the Cascade block.



$$T.F = \frac{\dfrac{G_1G_2G_3}{1+G_2H_1}}{1+\dfrac{G_2G_3H_2}{1+G_2H_1}}$$

$$T.F = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2}$$

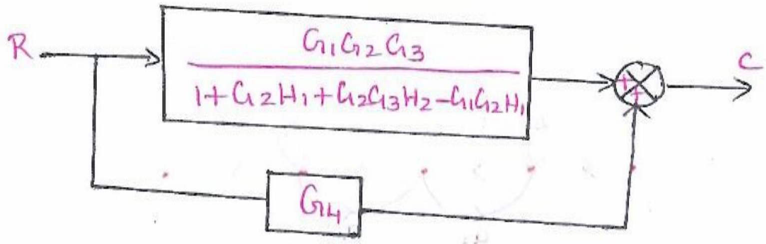
Step-7:- Eliminating the positive feedback path.



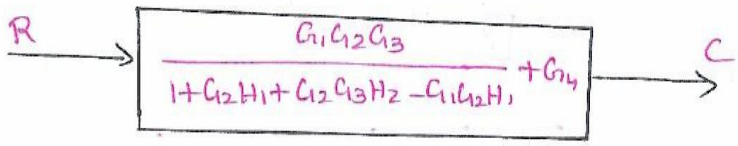
$$T.F = \frac{\dfrac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2}}{1-\dfrac{G_1G_2\cancel{G_3}}{1+G_2H_1+G_2G_3H_2}\cdot\dfrac{H_1}{\cancel{G_3}}}$$

$$T.F = \frac{\dfrac{G_1 G_2 G_3}{1+G_2H_1+G_2G_3H_2}}{\dfrac{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}{1+G_2H_1+G_2G_3H_2}}$$

$$T.F = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}$$



Step-8:- Combining the Parallel block.



$$\frac{C}{R} = \frac{G_1G_2G_3 + G_4(1+G_2H_1+G_2G_3H_2-G_1G_2H_1)}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}$$

$$\boxed{\frac{C}{R} = \frac{G_1G_2G_3 + G_4 + G_2G_4H_1 + G_2G_3G_4H_2 - G_1G_2G_4H_1}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}}$$

5) Draw the signal flow graph for the following system and obtain the transfer function using Mason's Gain Formula. [M/J - 2014]

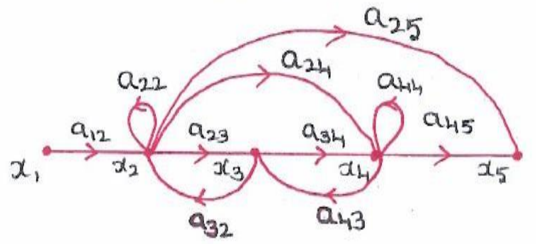
$$x_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$$

$$x_3 = a_{23}x_2 + a_{43}x_4$$

$$x_4 = a_{24}x_2 + a_{34}x_3 + a_{44}x_4$$

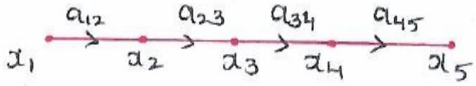
$$x_5 = a_{25}x_2 + a_{45}x_4.$$

Solution:-

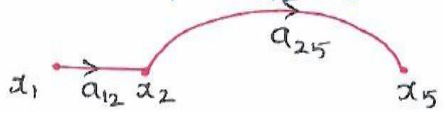


Step-1:- Forward path gain

$$\boxed{K=3}$$



$$P_1 = a_{12}a_{23}a_{34}a_{45}$$

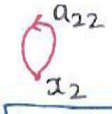


$$P_2 = a_{12}a_{25}$$

$$P_3 = a_{12}a_{24}a_{45}$$

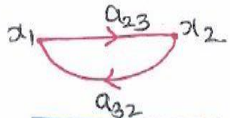
Step-2:- Individual Loop Gain

Loop-1



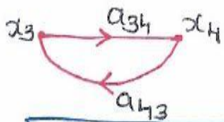
$$\boxed{L_1 = a_{22}}$$

Loop-2



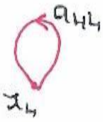
$$\boxed{L_2 = a_{32}a_{23}}$$

Loop-3



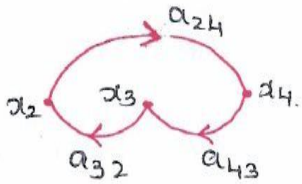
$$\boxed{L_3 = a_{34}a_{43}}$$

Loop-4



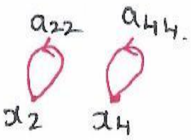
$$\boxed{L_4 = a_{44}}$$

Loop-5

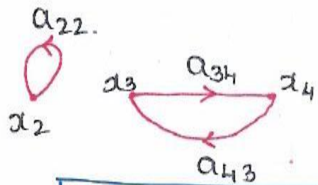


$$\boxed{L_5 = a_{24}a_{32}a_{43}}$$

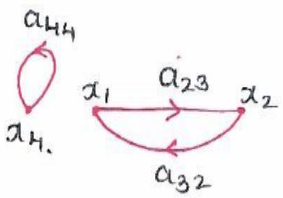
Step-3:- Gain product of two non-touching loops



$$\boxed{P_{12} = a_{22}a_{44}}$$



$$\boxed{P_{22} = a_{22}a_{34}a_{43}}$$



$$\boxed{P_{32} = a_{44}a_{23}a_{32}}$$

Step-4:- Transfer Function.

By Mason's Gain Formula,

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$T = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} \rightarrow ①$$

$$\Delta_1 = 1 - 0 = 1$$

$$\Delta_2 = 1 - [L_3 + L_4] = 1 - a_{34}a_{43} - a_{44}$$

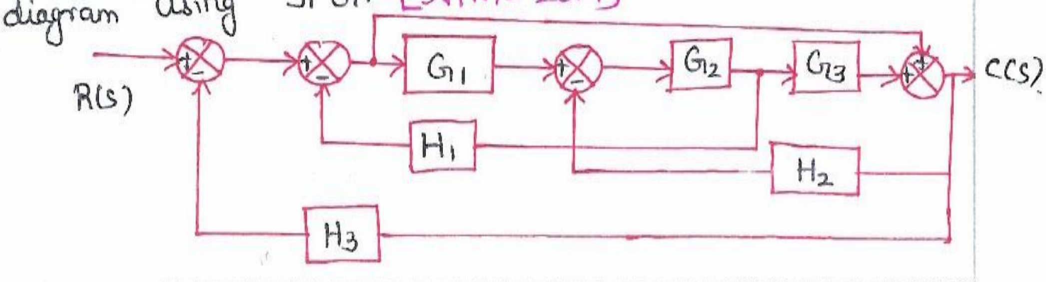
$$\Delta_3 = 1 - 0 = 1$$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + P_{12} + P_{22} + P_{32}$$

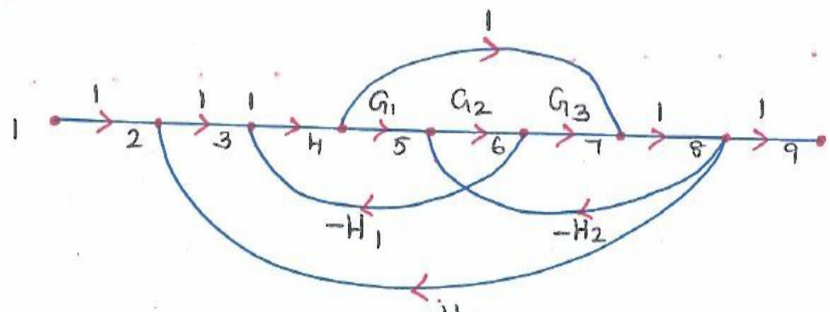
$$\Delta = 1 - a_{22} - a_{32}a_{23} - a_{34}a_{43} - a_{44} - a_{24}a_{32}a_{43} - a_{22}a_{44} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44}$$

$$① \Rightarrow \boxed{T = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{25} - a_{12}a_{25}a_{34}a_{43} - a_{12}a_{25}a_{44} - a_{12}a_{24}a_{44}}{1 - a_{22} - a_{23}a_{32} - a_{34}a_{43} - a_{44} - a_{24}a_{32}a_{43} - a_{22}a_{44} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44}}}$$

6) Find the overall gain of the transfer function of the system is represented by the block diagram using SFG. [A/M-2014]

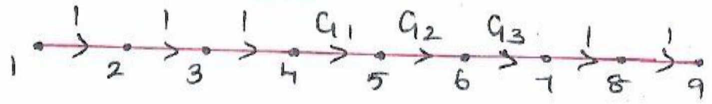


Solution:-

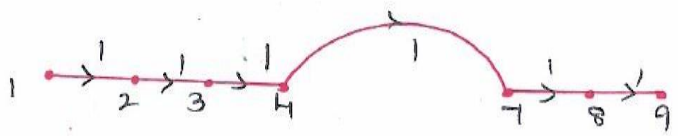


Step-1:- Forward Path gain.

$\boxed{K = 2}$



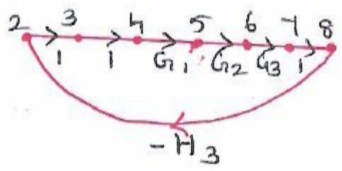
$$\boxed{P_1 = G_1 G_2 G_3}$$



$$\boxed{P_2 = 1}$$

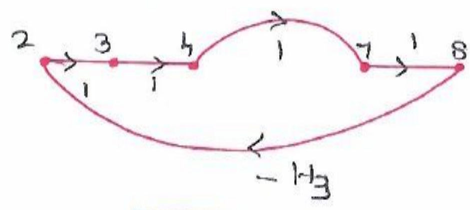
Step-2:- Individual loop gain.

Loop-1



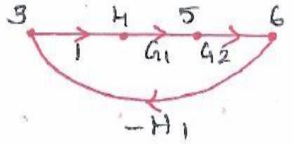
$\boxed{L_1 = -G_1 G_2 G_3 H_3}$

Loop-2



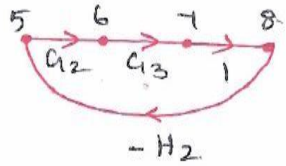
$\boxed{L_2 = -H_3}$

Loop-3



$$\boxed{L_3 = -G_1 G_2 H_1}$$

Loop-4



$$\boxed{L_4 = -G_2 G_3 H_2}$$

Step-3:- Gain product of two non-touching loops.

There is no two non-touching loops.

Step-4:- Transfer Function.

By Mason's Gain Formula,

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$T = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} \rightarrow (1)$$

$$\Delta_1 = 1 - 0 = 1$$

$$\Delta_2 = 1 - 0 = 1$$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4)$$

$$(1) \Rightarrow \boxed{T = \frac{1 + G_1 G_2 G_3}{1 + G_1 G_2 G_3 H_3 + G_1 G_2 H_1 + G_2 G_3 H_2}}$$

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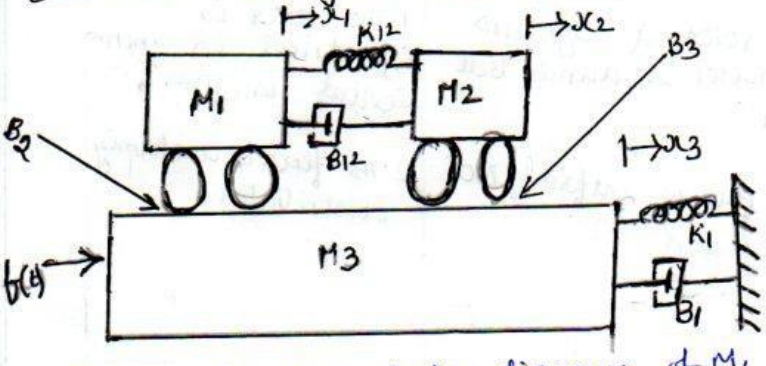
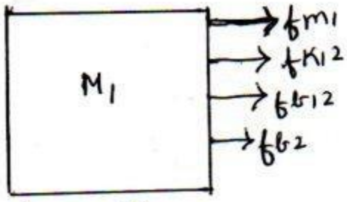
I

Part B

7. Write the differential equations governing mechanical system shown in fig. Draw force voltage & force current electrical analogous circuits? [NOV 16]



I Drawing free body diagram of M_1,



$0 = f_{m1} + f_{K12} + f_{b12} + f_{b2}$

$0 = B_2(v_1 - v_3) + M_1 \frac{d^2x_1}{dt^2} + B_{12}\frac{d(x_1-x_2)}{dt} + K_{12}(x_1-x_2)$ is force balance equation.

So, rewriting equation

$0 = B_2\frac{d(x_1-x_3)}{dt} + M_1\frac{dv_1}{dt} + B_{12}(v_1-v_2) + K_{12}\int(v_1-v_2)\,dt$ — (1)

Now FV analogous elements are,

$M_1 \rightarrow L_1$ $K_{12} \rightarrow \frac{1}{C_{12}}$

$v_1 \rightarrow i_1$ $B_2 \rightarrow R_2$

$B_{12} \rightarrow R_{12}$

$v_2 \rightarrow i_2$

So the analogous equation FV is,

$0 = R_2(i_1 - i_3) + L_1\frac{di_1}{dt} + R_{12}(i_1 - i_2) + \frac{1}{C_{12}}\int(i_1-i_2)\,dt$ — (1FV)

For this same equation FI analogous elements are,

$M_1 \to C_1$ $\quad K_{12} \to \frac{1}{L_{12}}$

$v_1 \to e_1$ $\quad B_2 \to \frac{1}{R_2}$

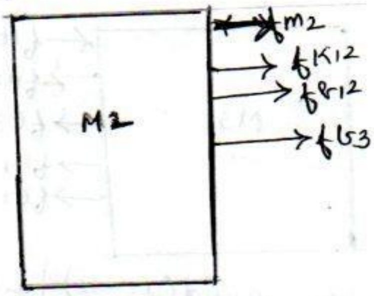
$B_{12} \to \frac{1}{R_{12}}$

$v_2 \to e_2$

So FI analogous equation is,

$$0 = \frac{1}{R_2}(e_1 - e_3) + C_1\frac{de_1}{dt} + \frac{1}{R_{12}}(e_1 - e_2) + \frac{1}{L_{12}}\int(e_1 - e_2)dt \quad \text{— (1 FI)}$$

II Drawing free body diagram of M_2,



$$0 = M_2\frac{d^2x_2}{dt^2} + K_{12}(x_2 - x_1) + B_{12}\frac{d(x_2 - x_1)}{dt} + B_3\frac{d(x_2 - x_3)}{dt}$$

$$0 = M_2\frac{dv_2}{dt} + K_{12}\int(v_2 - v_1)dt + B_{12}(v_2 - v_1) + B_3(v_2 - v_3) \quad \text{— (2)}$$

Writing FV analogous elements

$M_2 \to L_2$ $\quad B_3 \to R_3$

$v_2 \to i_2$ $\quad v_3 \to i_3$

$K_{12} \to \frac{1}{C_{12}}$

$v_1 \to i_1$

$B_{12} \to R_{12}$

So FV analogous element equation is

$$0 = L_2\frac{di_2}{dt} + \frac{1}{C_{12}}\int(i_2 - i_1)dt + R_{12}(i_2 - i_1) + R_3(i_2 - i_3) \quad \text{— (2 FV)}$$

FI analogous circuit elements are,

$M_2 \rightarrow C_2$ $\quad$ $B_3 \rightarrow \frac{1}{R_3}$

$v_2 \rightarrow e_2$ $\quad$ $v_3 \rightarrow e_3$

$K_{12} \rightarrow \frac{1}{L_2}$

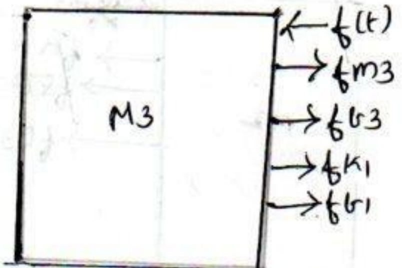
$v_1 \rightarrow e_1$

$B_{12} \rightarrow \frac{1}{R_{12}}$

So FI analogous equation is,

$$0 = C_2 \frac{de_2}{dt} + \frac{1}{L_2}\int (e_2 - e_1)dt + \frac{1}{R_{12}}(e_2 - e_1) + \frac{1}{R_3}(e_2 - e_3) \quad \text{— (2FI)}$$

Drawing free body diagram of M_3,



$$f(t) = M_3 \frac{d^2x_3}{dt^2} + B_3\left(\frac{d(x_3 - x_2)}{dt}\right) + K_1 x_3 + B_1 \frac{dx_3}{dt}$$

writing FV analogous elements,

$$f(t) = M_3 \frac{dv_3}{dt} + B_3(v_3 - v_2) + K_1\int v_3 dt + B_1 v_3 \quad \text{— (3)}$$

$f(t) \rightarrow e(t)$ $\quad$ $K_1 \rightarrow \frac{1}{C_1}$

$M_3 \rightarrow L_3$ $\quad$ $B_1 \rightarrow R_1$

$B_3 \rightarrow R_3$

$v_3 \rightarrow i_3$

$v_2 \rightarrow i_2$

$$e(t) = L_3 \frac{di_3}{dt} + R_3(i_3 - i_2) + \frac{1}{C_1}\int i_3 dt + R_1 i_3 \quad \text{— (3FV)}$$

Writing FI analogous elements,

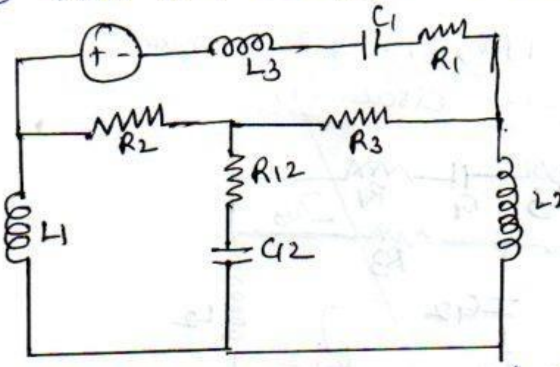
$f(t) \rightarrow i(t)$ $\quad$ $B_3 \rightarrow \frac{1}{R_3}$ $\quad$ $v_2 \rightarrow e_2$

$M_3 \rightarrow C_3$ $\quad$ $v_3 \rightarrow e_3$ $\quad$ $K_1 \rightarrow \frac{1}{L_1}$

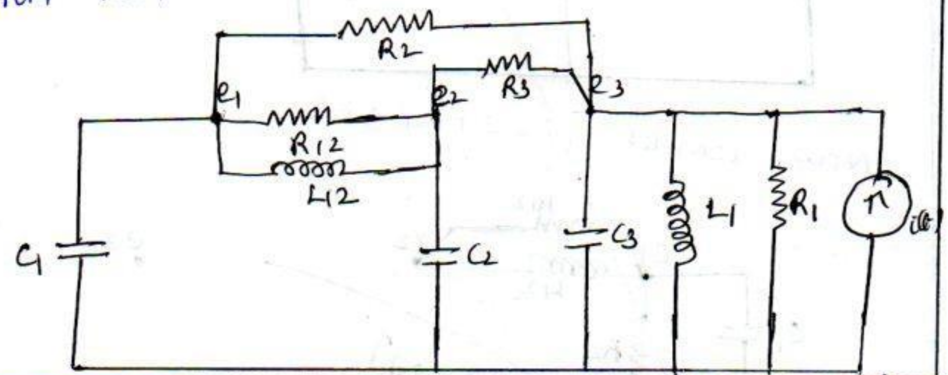
$B_1 \rightarrow 1/R_1$

$$i(t) = C_3\frac{de_3}{dt} + \frac{1}{R_3}(e_3 - e_2) + \frac{1}{L}\int e_3\,dt + \frac{1}{R_1}e_3 \quad -3FI$$

Now From 1FV, 2FV, 3FV FV analogous circuit is,

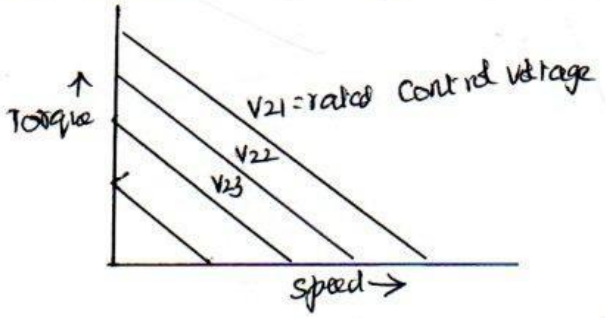


From 1FI, 2FI & 3FI FI analogous circuit is,



8. Derive transfer function of AC Servo motor? (Nov16)

Consider characteristics of AC servo motor, (Apr17)



Let $K = \dfrac{\text{Blocked rotor torque}}{\text{Rated phase control voltage}}$

$$m = \frac{-T_0}{N_0} = \frac{-\text{Rotor Torque}}{\text{No load speed}}$$

Static torque $= KV_2$

Dynamic torque $= m \dfrac{d\theta m}{dt}$

T_d = torque developed by motor

= Static Torque + Dynamic torque

$= KV_2 + m \dfrac{d\theta m}{dt}$

$$T_d = KV_2 + m\frac{d\theta m}{dt} = J\frac{d^2\theta m}{dt^2} + B\frac{d\theta m}{dt} \quad \text{— (1)}$$

Taking Laplace transform,

$$T_d(s) = KV_2(s) + m s\,\theta m(s) = Js^2\,\theta m(s) + Bs\,\theta m(s)$$

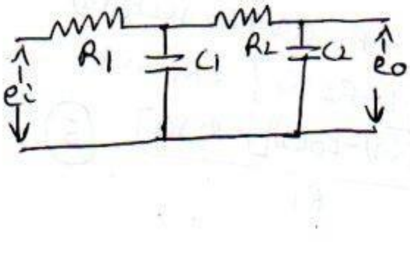
$$KV_2(s) = Js^2\,\theta m(s) + Bs\,\theta m(s) - ms\,\theta m(s)$$

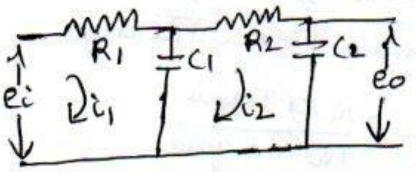
$$= \theta m(s)\left[Js^2 + Bs - ms\right]$$

$$\frac{\theta m(s)}{V_2(s)} = \left(\frac{Js^2 + Bs - ms}{K}\right)^{-1}$$

$$\frac{\theta m(s)}{V_2(s)} = \frac{K}{Js^2 + s(B - m)}$$

9. Construct a block diagram of the simple electrical network shown in fig. Hence obtain signal flow graph and transfer function $\dfrac{E_0(s)}{E_i(s)}$ [NOV16]





For Mesh I,

$$E_i(s) = \left(R_1 I_1(s) + \frac{1}{sC_1}\left(I_1(s) - I_2(s)\right)\right)$$

$$E_i(s) = R_1 I_1(s) + \frac{1}{sC_1}\left[I_1(s) - I_2(s)\right] \quad \text{— (1)}$$

$$= I_1(s)\left[R_1 + \frac{1}{sC_1}\right] - \frac{1}{sC_1} I_2(s) \quad \text{— (1a)}$$

$$E_o(s) = \frac{1}{sC_2} I_2(s) \quad \text{— (2)}$$

For Mesh II,

$$0 = \frac{1}{sC_1}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{sC_2} I_2(s) \quad \text{— (2a)}$$

$$= -I_1(s)\left[\frac{1}{sC_1}\right] + I_2(s)\left[\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}\right] \quad \text{— (2b)}$$

Substitute (2) in (2a)

$$0 = \frac{1}{sC_1}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + E_o(s)$$

$$\frac{1}{sC_1}\left[I_1(s) - I_2(s)\right] = R_2 I_2(s) + E_o(s) \quad \text{— (3)}$$

Substitute (3) in (1)

$$E_i(s) = R_1 I_1(s) + R_2 I_2(s) + E_o(s) \quad \text{— (4)}$$

So $E_i(s) - E_o(s) = R_1 I_1(s) + R_2 I_2(s)$

From (2) $I_2(s) = sC_2 E_o(s)$ So,

$$E_i(s) - E_o(s) = R_1 I_1(s) + R_2 sC_2 E_o(s)$$

$$E_i(s) - E_o(s)\left[1 - R_2 sC_2\right] = R_1 I_1(s)$$

$$I_1(s) = \frac{E_i(s) - E_o(s)\left[1 - R_2 sC_2\right]}{R_1} \quad \text{— (5)}$$

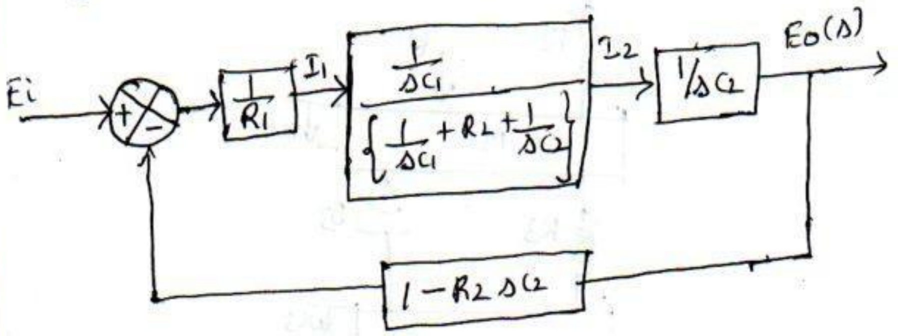
From (2b),

$$I_1(s)\left[\frac{1}{sC_1}\right] = I_2(s)\left[\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}\right]$$

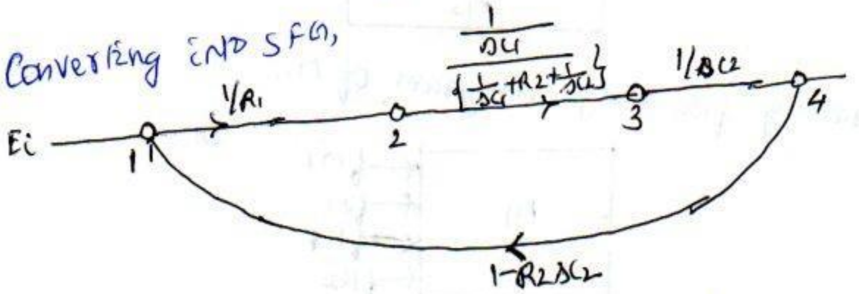
$$\text{So, } I_2(s) = \frac{I_1(s)\left[\frac{1}{sC_1}\right]}{\left[\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}\right]} \quad \text{—} (6)$$

$$E_o(s) = \frac{1}{sC_2} I_2(s) \quad \text{—} (7)$$

By (5), (6) & (7) We construct Block diagram,



Converting into SFG,



Forward path:

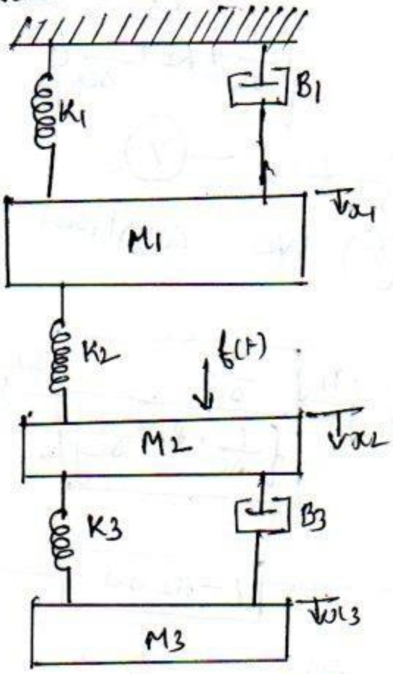
1-2-3-4 of gain $\left\{\frac{1}{R_1}\right\}\left\{\frac{\frac{1}{sC_1}}{\left\{\frac{1}{sC_1} + R_1 + \frac{1}{sC_2}\right\}}\right\}$.

Individual loop

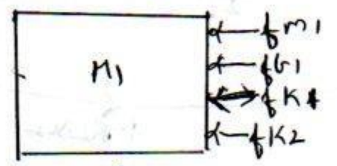
1-2-3-4-1 of gain $\left\{\frac{1}{R_1}\right\}\left\{\frac{\frac{1}{sC_1}}{\left\{\frac{1}{sC_1} + R_1 + \frac{1}{sC_2}\right\}}\right\}(1 - R_2 sC_2)$

$$TF = \frac{\sum P_K \Delta_K}{\Delta} = \frac{\frac{1}{R_1}\left\{\frac{\frac{1}{sC_1}}{\left\{\frac{1}{sC_1} + R_1 + \frac{1}{sC_2}\right\}}\right\}(1)}{1 - \frac{1}{R_1}\left\{\frac{1/sC_1}{\frac{1}{sC_1} + R_1 + \frac{1}{sC_2}}\right\}\{1 - R_2 sC_2\}}$$

10. Write the differential equations governing mechanical translational system shown in fig & draw electrical analogous circuits? [Apr 17]



Drawing free body diagram of M_1,



Force balance equation is,

$$0 = M_1\frac{d^2x_1}{dt^2} + K_1x_1 + B_1\frac{dx_1}{dt} + K_2(x_1 - x_2)$$

$$(or)\quad 0 = M_1\frac{dv_1}{dt} + K_1\int v_1\,dt + B_1v_1 + K_2\int(v_1 - v_2)\,dt \quad \text{--- (1)}$$

Writing FV analogous equation,

$M_1 \rightarrow L_1 \qquad B_1 \rightarrow R_1$

$K_1 \rightarrow \frac{1}{C_1} \qquad K_2 \rightarrow \frac{1}{C_2}$

$v_1 \rightarrow i_1 \qquad v_2 \rightarrow i_2$

$$0 = L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + R_1 i_1 + \frac{1}{C_2}\int (i_1 - i_2)\, dt \quad \text{— (1FV)}$$

Writing FI analogous circuit

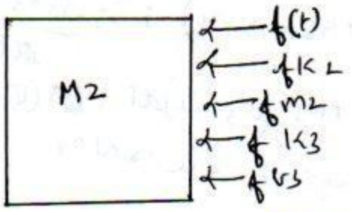
$M_1 \to C_1$ $\quad B_1 \to \frac{1}{R_1}$

$K_1 \to \frac{1}{L_1}$ $\quad K_2 \to \frac{1}{L_2}$

$v_1 \to e_1$ $\quad v_2 \to e_2$

$$0 = C_1 \frac{de_1}{dt} + \frac{1}{L_1}\int e_1 dt + \frac{1}{R_1} e_1 + \frac{1}{L_2}\int (e_1 - e_2)\, dt \quad \text{— (1FI)}$$

So we find free body diagram of M_2,



$$f(t) = M_2 \frac{d^2 x_2}{dt^2} + K_2 (x_2 - x_1) + K_3 (x_2 - x_3) + B_3 \frac{d(x_2 - x_3)}{dt}$$

$$= M_2 \frac{dv_2}{dt} + K_2 \int (v_2 - v_1)\, dt + K_3 \int (v_2 - v_3)\, dt + B_3 (v_2 - v_3) \quad \text{— (2)}$$

Now from FV Analogous circuit

$M_2 \to L_2$ $\quad K_3 \to \frac{1}{C_3}$

$v_2 \to i_2$ $\quad v_3 \to i_3$

$K_2 \to \frac{1}{C_2}$ $\quad B_3 \to R_3$

$v_1 \to i_1$ $\quad f(t) \to e(t)$

So, $$e(t) = L_2 \frac{di_2}{dt} + \frac{1}{C_2}\int (i_2 - i_1)\, dt + \frac{1}{C_3}\int (i_2 - i_3)\, dt + R_3 (i_2 - i_3) \quad \text{— (2FV)}$$

Now from FI analogous circuit,

$M_2 \to C_2$ $\quad K_3 \to \frac{1}{L_3}$

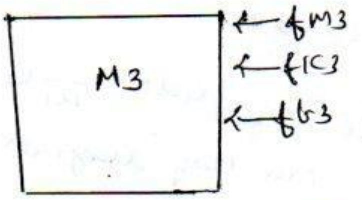
$v_2 \to e_2$ $\quad v_3 \to e_3$

$K_2 \to \frac{1}{L_2}$ $\quad B_3 \to \frac{1}{R_3}$

$v_1 \to e_1$ $\quad f(t) \to i(t)$

$$i(t) = C_2 \frac{de_2}{dt} + \frac{1}{L_2}\int (e_2 - e_1)\,dt + \frac{1}{L_3}\int (e_2 - e_3)\,dt + \frac{1}{R_3}(e_2 - e_3) \quad \text{— (2 FI)}$$

Drawing free body diagram of M_3,



$$0 = M_3 \frac{d^2 x_3}{dt^2} + K_3 (x_3 - x_2) + B_3 \frac{d(x_3 - x_2)}{dt}$$

$$\text{(or)}\quad 0 = M_3 \frac{dv_3}{dt} + K_3 \int (v_3 - v_2)\,dt + B_3 (v_3 - v_2) \quad \text{— (3)}$$

Writing FV analogous elements,

$M_3 \rightarrow L_3$
$v_3 \rightarrow i_3$
$K_3 \rightarrow \frac{1}{C_3}$
$v_2 \rightarrow i_2$
$B_3 \rightarrow R_3$

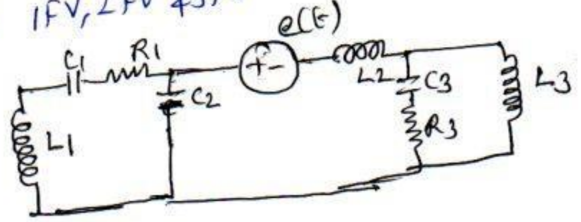
So $$0 = L_3 \frac{di_3}{dt} + \frac{1}{C_3}\int (i_3 - i_2)\,dt + R_3 (i_3 - i_2) \quad \text{— (3 FV)}$$

Writing FI analogous elements,

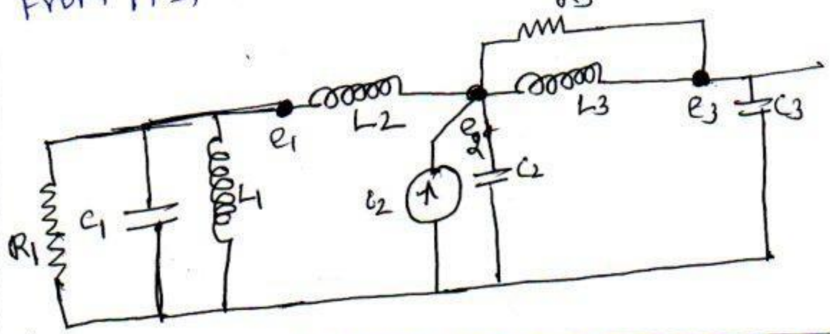
$M_3 \rightarrow C_3$
$v_3 \rightarrow e_3$
$K_3 \rightarrow \frac{1}{L_3}$
$v_2 \rightarrow e_2$
$B_3 \rightarrow 1/R_3$

So $$0 = C_3 \frac{de_3}{dt} + \frac{1}{L_3}\int (e_3 - e_2)\,dt + \frac{1}{R_3}(e_3 - e_2) \quad \text{— (3 FI)}$$

From 1FV, 2FV & 3FV



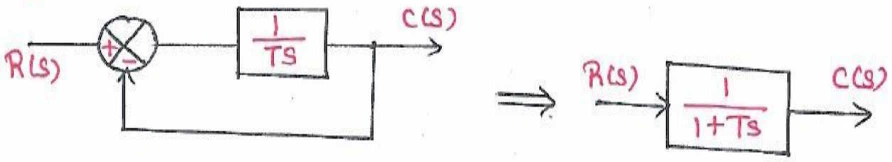
From 1FI, 2FI & 3FI



PART-B

1)a) Discuss the unit step response of first order system. [N/D-2015]

The closed loop order system with unity feedback is shown below,



The closed loop transfer function of first order system, $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

If the unit step input is applied, $r(t)=1$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s(1+Ts)}$$

$$C(s) = \frac{1}{sT(s+1/T)}$$

By partial fraction,

$$C(s) = \frac{A}{s} + \frac{B}{s+1/T}$$

A is obtained by multiplying C(s) by s and letting s=0

$$A = C(s) \times s \Big|_{s=0}$$

$$A = \frac{1/T}{S(S+1/T)} S \Bigg|_{S=0} = \frac{1/T}{1/T}$$

$$\boxed{A=1}$$

B is obtained by multiplying C(s) by (S+1/T) and letting S=-1/T

$$B = C(S) \times \left(S+\frac{1}{T}\right)\Bigg|_{S=-\frac{1}{T}}$$

$$B = \frac{\frac{1}{T}}{S\cancel{\left(S+\frac{1}{T}\right)}} \times \cancel{\left(S+\frac{1}{T}\right)}\Bigg|_{S=-\frac{1}{T}}$$

$$B = \frac{1/T}{-1/T}$$

$$\boxed{B=-1}$$

$$C(S) = \frac{1}{S} - \frac{1}{S+1/T}$$

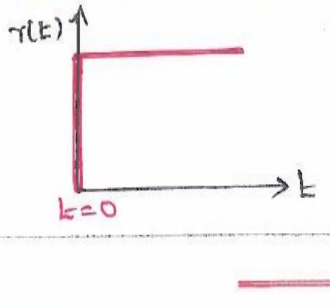
Taking inverse laplace transform,

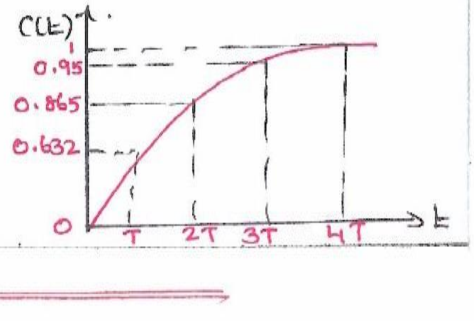
$$\boxed{C(t) = 1 - e^{-t/T}}$$

When, $t=0$, $C(t)=0$

when, $t=\infty$, $C(t)=1$

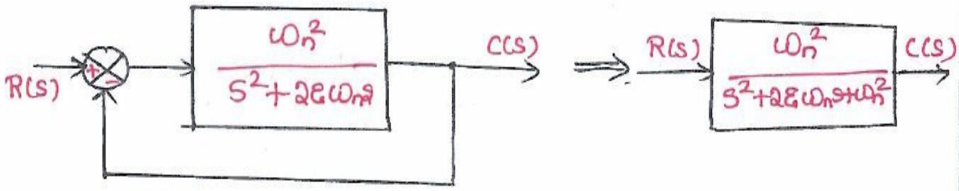
The response for the first order system with unit-step input is shown below.





b) Derive the expression with unit step second order system for undamped and underdamped system. [A/M-2010, A/M-2011, M/J-2013, A/M-2015, N/D-2016]

The closed loop second order system is shown below,



The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

where, $\omega_n \rightarrow$ Undamped natural frequency.
$\xi \rightarrow$ Damping ratio.

The damping ratio is defined as the ratio of actual damping to the critical damping. Depending on the value of ξ, the system can be classified into four cases.

Case 1: Undamped system [$\xi = 0$]

Case 2: Underdamped system. [$0 < \xi < 1$]

Case 3: Critically damped system $[\xi = 1]$

Case 4: Overdamped system. $[\xi > 1]$

The characteristics equation of second order system is,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \rightarrow (1)$$

The roots of this equation is,

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2-1)}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} \rightarrow (2)$$

When $\xi = 0$, $s_1, s_2 = \pm j\omega_n$ $\begin{cases} \text{roots are Purely imaginary} \\ \text{and the System is undamped} \end{cases}$

When $\xi = 1$, $s_1, s_2 = -\omega_n$ $\begin{cases} \text{roots are real and equal and} \\ \text{the System is critically damped} \end{cases}$

When $\xi > 1$, $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ $\begin{cases} \text{roots are real and unequal} \\ \text{and the system is overdamped} \end{cases}$

When $0 < \xi < 1$, $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} = -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -\xi\omega_n \pm j\omega_d \begin{cases} \text{roots are Complex Conjugate} \\ \text{and system is underdamped.} \end{cases}$$

Where, $\omega_d = \omega_n\sqrt{1-\xi^2}$. (damping frequency)

Response of Undamped Second order System for Unit Step input.

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For undamped system, $\xi = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2}$$

For unit-step input, $r(t)=1$ and $R(s)=\frac{1}{s}$

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

By partial fraction,

$$C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2}.$$

A is obtained by multiplying $C(s)$ by s and letting $s=0$.

$$A = C(s)\times s\Big|_{s=0} = \frac{\omega_n^2}{\cancel{s}(s^2+\omega_n^2)}\cancel{s}\Bigg|_{s=0}$$

$$A = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying $C(s)$ by $(s^2+\omega_n^2)$ and letting $s^2 = -\omega_n^2$ (or) $s = -j\omega_n$

$$B = C(s) \times (s^2+\omega_n^2) \Big|_{s=j\omega_n}$$

$$B = \frac{\omega_n^2}{s(\cancel{s^2+\omega_n^2})} (\cancel{s^2+\omega_n^2}) \Big|_{s=j\omega_n}$$

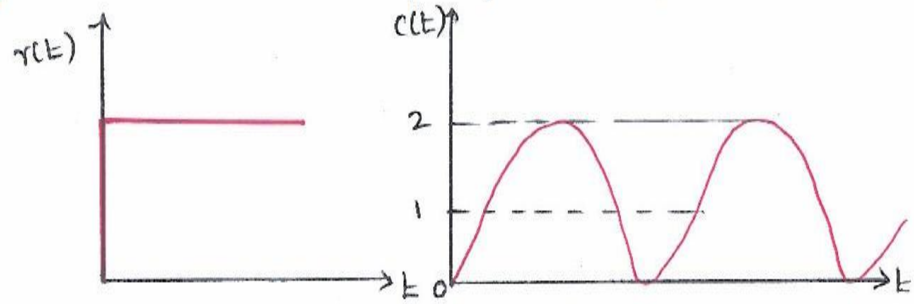
$$B = \frac{\omega_n^{\cancel{2}}}{j\cancel{\omega_n}} = -j\omega_n = -s.$$

$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

Taking inverse laplace transform,

$$\boxed{c(t) = 1 - \cos\omega_n t}$$

The response for undamped second order system with unit-step input is shown below.



Response of Underdamped Second order System with unit-step input.

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For underdamped system, $0 < \xi < 1$,

For unit-step input $r(t) = 1$ and $R(s) = \frac{1}{s}$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

A is obtained by multiplying $C(s)$ by s and letting $s=0$

$$A = C(s) \times s \Big|_{s=0} = \frac{\omega_n^2}{\cancel{s}(s^2+2\xi\omega_n s+\omega_n^2)} \times \cancel{s} \Big|_{s=0}$$

$$A = \frac{\omega_n^2}{\omega_n^2} = 1$$

To find B and C,

$$C(s) = \frac{1}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\omega_n^2 = s^2+2\xi\omega_n s+\omega_n^2 + Bs^2+Cs$$

Equating the co-efficients of s^2,

$$0 = 1+B$$

$$\boxed{B=-1}$$

Equating the co-efficients of s,

$$0 = 2\xi\omega_n + C$$

$$\boxed{C = -2\xi\omega_n}$$

$$C(s) = \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{s^2+2\varepsilon\omega_n s+\omega_n^2}$$

Add and subtract $\varepsilon^2\omega_n^2$ to denominator of second term,

$$C(s) = \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{s^2+2\varepsilon\omega_n s+\varepsilon^2\omega_n^2-\varepsilon^2\omega_n^2+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{(s^2+2\varepsilon\omega_n s+\varepsilon^2\omega_n^2)+(\omega_n^2-\varepsilon^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_n^2(1-\varepsilon^2)}$$

$$C(s) = \frac{1}{s} - \frac{s+\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_d^2} - \frac{\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_d^2}$$

Let us multiply and divide by ω_d in the third term,

$$C(s) = \frac{1}{s} - \frac{s+\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_d^2} - \frac{\varepsilon\omega_n}{\omega_d}\frac{\omega_d}{(s+\varepsilon\omega_n)^2+\omega_d^2}$$

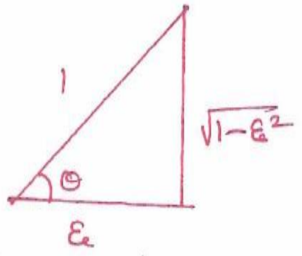
Taking inverse laplace transform,

$$c(t) = 1 - e^{-\varepsilon\omega_n t}\cos\omega_d t - \frac{\varepsilon\omega_n}{\omega_d}e^{-\varepsilon\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\varepsilon\omega_n t}\left[\cos\omega_d t + \frac{\varepsilon\omega_n}{\omega_n\sqrt{1-\varepsilon^2}}\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}}\left[\sqrt{1-\varepsilon^2}\cos\omega_d t + \varepsilon\sin\omega_d t\right]$$

Consider a right angled-triangle,



$$\sin\theta = \sqrt{1-\varepsilon^2}$$

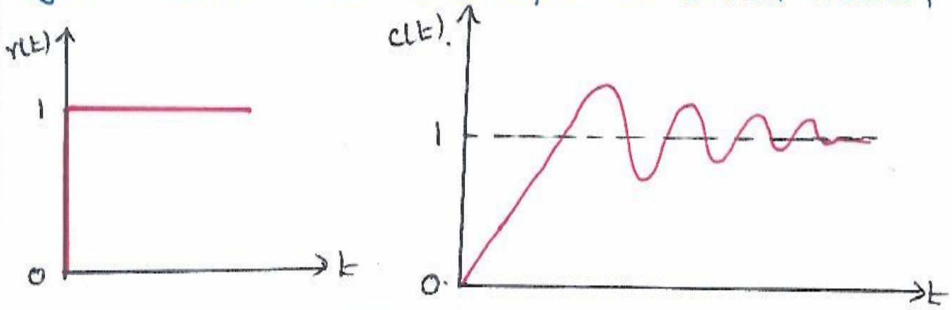
$$\cos\theta = \varepsilon$$

$$\tan\theta = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}.$$

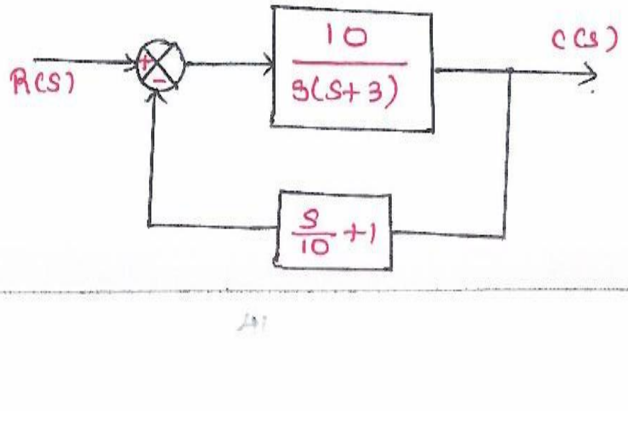
$$C(t) = 1 - \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}}\left[\sin\omega_d t\cos\theta + \cos\omega_d t\sin\theta\right]$$

$$\boxed{C(t) = 1 - \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}}\sin(\omega_d t+\theta)}$$

The response of under-damped second-order system with unit step input is shown below,



2)a) Determine the unit-step response of the control system shown in figure. [M|J-2014]



Given:-

$$G(s) = \frac{10}{s(s+3)}, \quad H(s) = 0.1s+1, \quad R(s) = 1$$

To find:-

$C(t)$.

Solution:-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{10}{s(s+3)}}{1+\frac{10}{s(s+3)}(0.1s+1)}$$

$$= \frac{\frac{10}{s\cancel{(s+3)}}}{\frac{s(s+3)+10(0.1s+1)}{s\cancel{(s+3)}}}$$

$$= \frac{10}{s^2+3s+1s+10}$$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+4s+10}$$

$$C(s) = \frac{10}{s(s^2+4s+10)}$$

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+4s+10}$$

$$10 = A(s^2+4s+10) + s(Bs+C)$$

$$10 = A(s^2+4s+10) + Bs^2+Cs.$$

Sub, $s=0$,

$$10 = 10A \Rightarrow \boxed{A=1}$$

Equating co-efficient of s^2,

$$0 = A + B$$

$$\boxed{B = -1}$$

Equating co-efficient of s,

$$0 = 4A + C$$

$$\boxed{C = -4}$$

$$C(s) = \frac{1}{s} - \frac{s+4}{s^2+4s+10}$$

$$= \frac{1}{s} - \frac{s+4}{s^2+4s+4+6}$$

$$= \frac{1}{s} - \frac{s+4}{(s+2)^2+6}$$

$$C(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+6} - \frac{2}{(s+2)^2+6}$$

Multiply & divide by $\sqrt{6}$ in 3^{rd} term,

$$C(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+\sqrt{6}^2} - \frac{2\sqrt{6}}{\sqrt{6}} \frac{1}{(s+2)^2+\sqrt{6}^2}$$

Taking inverse laplace transform,

$$\boxed{c(t) = 1 - e^{-2t}\cos\sqrt{6}\,t - \frac{2}{\sqrt{6}}\sin\sqrt{6}\,t}$$

b) A unity feedback system is characterized by open loop TF $G(s) = \frac{K}{s(s+10)}$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input and settling time. [H|J-14, A|M-11, N|D-12]

Given:-

$$G(s) = \frac{K}{s(s+10)}, \quad H(s) = 1, \quad \xi = 0.5$$

To find:-

Settling time, t_s

Peak time, t_p

Peak overshoot, $\% M_p$

Gain, K.

Solution:-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}(1)}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+10)+K} = \frac{K}{s^2+10s+K} \rightarrow (1)$$

We know that,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \rightarrow (2)$$

comparing eqn (1) & (2),

$$\omega_n^2 = K$$

$$2\xi\omega_n = 10$$

$$\omega_n = \frac{10}{2\xi} = \frac{10}{2\times 0.5}$$

$$\boxed{\omega_n = 10}$$

Gain, $\boxed{K = \omega_n^2 = 100}$

Settling time, t_s,

For 2%, $t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.5\times 10}$

$$\boxed{t_s = 0.8\ sec}$$

For 5%, $t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.5\times 10}$

$$\boxed{t_s = 0.6\ sec}$$

Peak time,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-0.5^2}}$$

$$\boxed{t_p = 0.362\ secs}$$

Peak Overshoot,

$$\% M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100$$

$$\boxed{\% M_p = 16.3\%}$$

3) a) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(sT+1)}$, where K and T are positive constant. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%. [N/D-10, M/J-12]

Given:-

$$G(s) = \frac{K}{s(sT+1)}, \quad H(s) = 1, \quad \% M_p = 75\% \text{ to } 25\%$$

To find:-

Gain, K.

Solution:-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{K/s(sT+1)}{1+\frac{K}{s(sT+1)}(1)}$$

$$= \frac{K}{s^2T + s + K}$$

$$\frac{C(s)}{R(s)} = \frac{K/T}{s^2 + \frac{s}{T} + \frac{K}{T}} \rightarrow (1)$$

We know that,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \rightarrow (2)$$

Comparing eqn (1) & (2),

$$\omega_n^2 = \frac{K}{T}$$

$$2\xi\omega_n = \frac{1}{T}$$

$$\xi = \frac{1}{2\omega_n T}$$

$$\xi = \frac{1}{2\sqrt{\frac{K}{T}} \times T}$$

$$\boxed{\xi = \frac{1}{2\sqrt{KT}}}$$

Peak overshoot, $M_p = e^{-\xi\pi / \sqrt{1-\xi^2}}$

Taking 'ln' on both sides,

$$\ln(M_p) = \frac{-\xi\pi}{\sqrt{1-\xi^2}}$$

Squaring on both sides,

$$[\ln(M_p)]^2 = \frac{\xi^2\pi^2}{1-\xi^2}$$

$$(1-\xi^2)\left[\ln(M_p)\right]^2 = \xi^2\pi^2$$

$$\left[\ln(M_p)\right]^2 - \xi^2(\ln M_p)^2 = \xi^2\pi^2$$

$$(\ln M_p)^2 = \xi^2\pi^2 + \xi^2(\ln M_p)^2$$

$$(\ln M_p)^2 = \xi^2\left[\pi^2 + (\ln M_p)^2\right]$$

$$\xi^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2} \rightarrow (1)$$

But, $\xi = \frac{1}{2\sqrt{KT}}$, $\xi^2 = \frac{1}{4KT} \rightarrow (2)$

$$\frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

$$\frac{1}{K} = \frac{4T(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

$$K = \frac{\pi^2 + (\ln M_p)^2}{4T(\ln M_p)^2}$$

When $K = K_1$, $M_p = 0.75$

$$K_1 = \frac{\pi^2 + (\ln 0.75)^2}{4T(\ln 0.75)^2} = \frac{9.952}{0.331T}$$

$$\boxed{K_1 = \frac{30.06}{T}}$$

When $K = K_2$, $M_p = 0.25$

$$K_2 = \frac{\pi^2 + [\ln(0.25)]^2}{4T(\ln 0.25)^2} = \frac{11.79}{7.687}$$

$$\boxed{K_2 = \frac{1.53}{T}}$$

$$\frac{K_1}{K_2} = \frac{(1/T)(30.06)}{(1/T)(1.53)}$$

$$\frac{K_1}{K_2} = 19.6 \Rightarrow K_1 = 19.6 K_2 \text{ (or) } K_2 = \frac{1}{19.6} K_1$$

To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times.

Gain, $\boxed{K = 20.}$

b) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{20}{s(s+2)}$. The input function is $r(t) = 2 + 3t + t^2$. Determine the generalised error co-efficient and steady-state error. [M-J/2014]

Given:-

$$G(s) = \frac{20}{s(s+2)}, \quad H(s) = 1, \quad r(t) = 2 + 3t + t^2$$

To find:-

C_0, C_1, C_2, e_{ss}.

Solution:-

$$r'(t) = 3+2t$$

$$r''(t) = 2$$

$$r'''(t) = 0$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{20}{s(s+2)}(1)} = \frac{s(s+2)}{s(s+2)+20}$$

$$F(s) = \frac{s^2+2s}{s^2+2s+20}$$

$$C_0 = \underset{s\to 0}{Lt}\, F(s) = \underset{s\to 0}{Lt}\, \frac{s^2+2s}{s^2+2s+20}$$

$$\boxed{C_0 = 0}$$

$$C_1 = \underset{s\to 0}{Lt}\, \frac{dF(s)}{ds} = \underset{s\to 0}{Lt}\, \frac{d}{ds}\left[\frac{s^2+2s}{s^2+2s+20}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^2+2s+20)(2s+2)-(s^2+2s)(2s+2)}{(s^2+2s+20)^2}\right]$$

$$= \frac{20(2)-2(0)}{20^2}$$

$$\boxed{C_1 = 0.1}$$

$$C_2 = \underset{s\to 0}{Lt}\, \frac{d^2F(s)}{ds^2} = \underset{s\to 0}{Lt}\, \frac{d}{ds}\left[\frac{2s^3+2s^2+4s^2+4s+40s+40-2s^3-2s^2-4s^2-4s}{(s^2+2s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\, \frac{d}{ds}\left[\frac{40s+40}{(s^2+2s+20)^2}\right]$$

$$= \underset{s \to 0}{Lt} \left[\frac{40(s^2+2s+20)^2(1) - (s+1)\,2(2s+2)}{(s^2+2s+20)^4} \right]$$

$$= \frac{40(20^2-4)}{20^4}$$

$$\boxed{C_2 = 0.08}$$

$$e(t) = r(t)C_0 + r'(t)C_1 + \frac{r''(t)C_2}{2!}$$

$$e(t) = 0 + (3+2t)(0.1) + \frac{\cancel{2} \times 0.08}{\cancel{2}}$$

$$e(t) = 0 + 0.3 + 0.2t + 0.08$$

$$\boxed{e(t) = 0.2t + 0.38}$$

Steady-state error,

$$e_{ss} = \underset{t \to \infty}{Lt}\ e(t)$$

$$e_{ss} = \underset{t \to \infty}{Lt}\ (0.2t + 0.38)$$

$$\boxed{e_{ss} = \infty}$$

Result:-

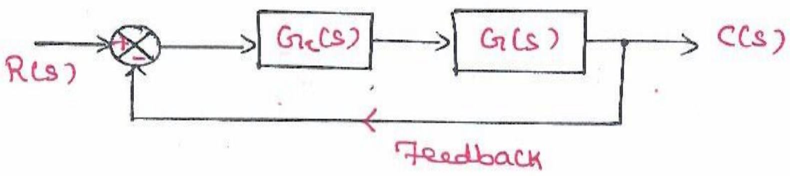
$C_0 = 0$
$C_1 = 0.1$
$C_2 = 0.08$
$e_{ss} = \infty$

4) Explain the effect by adding P, PI and PID Controller in feedback control System. [A/M-11, N/D-10, A/M-11, N/D-14]

Controller:-

A controller is a device which when introduced in feedback or forward path system, controls the steady state and transient response as per the requirement.

P-Controller:- [Proportional Controller]



The proportional controller is a device that produces a control signal u(t) which is proportional to the error signal.

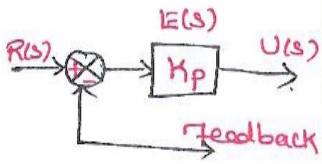
$$u(t) \propto e(t)$$

$$u(t) = K_p e(t)$$

Taking laplace transform,

$$U(s) = K_p E(s)$$

$$\boxed{\frac{U(s)}{E(s)} = K_p.}$$



Effect of P-controller:-

→ The P-controller produces an output signal which is proportional to error signal.

→ The P-controller is used to increase the loop gain of the system by following ways:-

a) Steady-state tracking accuracy.

b) Disturbance signal rejection

c) Relative stability.

→ The main drawback of P-controller is, it produces constant steady state error.

PI-Controller:- [Proportional Plus Integral Controller]

The PI controller produces an output signal consisting of two terms.

a) Proportional to error signal

b) Proportional to integral of error signal

In PI-controller,

$$u(t) \propto \left[e(t) + \int e(t)\,dt\right]$$

$$u(t) = K_p e(t) + \frac{K_p}{T_i} \int e(t) \cdot dt.$$

Taking laplace transform,

$$U(s) = K_p E(s) + \frac{K_p}{T_i s} E(s)$$

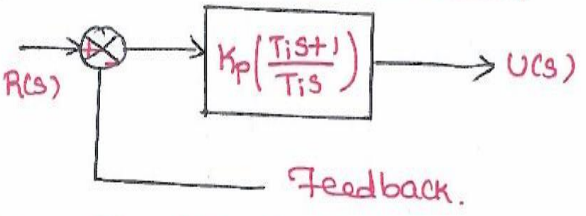
$$U(s) = E(s)\left[K_p + \frac{K_p}{T_i s}\right]$$

$$\frac{U(s)}{E(s)} = K_p\left[1 + \frac{1}{T_i s}\right]$$

$$\boxed{\frac{U(s)}{E(s)} = K_p\left[\frac{T_i s + 1}{T_i s}\right]}$$

$T_i \rightarrow$ Time constant

$K_p \rightarrow$ Proportional gain.



Advantages of PI - Controller:-

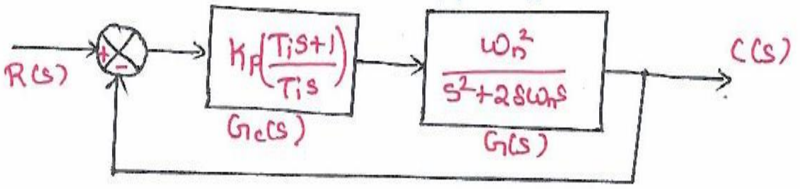
→ Increases the loop gain

→ Reduces the steady state error.

Effect of PI - Controller:-

The transfer function of PI - controller is,

$$G_c(s) = K_p\left(\frac{T_i s + 1}{T_i s}\right).$$



$$G(s)_{new} = G_c(s) \cdot G(s)$$

$$G(s)_{new} = K_p\left(\frac{T_i s+1}{T_i s}\right)\left(\frac{\omega_n^2}{s^2+2\delta\omega_n s}\right)$$

$$= \frac{\omega_n^2 K_p (T_i s + 1)}{T_i s (s^2 + 2\delta\omega_n s)}$$

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)_{new}}{1+G(s)H(s)}$$

$$= \frac{\dfrac{\omega_n^2 K_p(1+T_i s)}{s(s^2+2\delta\omega_n s)T_i}}{1+\dfrac{\omega_n^2 K_p(1+T_i s)}{s(s^2+2\delta\omega_n s)T_i}\times 1}$$

$$= \frac{\dfrac{\omega_n^2 K_p(1+T_i s)}{\cancel{s(s^2+2\delta\omega_n s)T_i}}}{\dfrac{s(s^2+2\delta\omega_n s)T_i + \omega_n^2 K_p(1+T_i s)}{\cancel{s(s^2+2\delta\omega_n s)T_i}}}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 K_p(1+T_i s)}{s^3 T_i + 2\delta\omega_n s^2 T_i + \omega_n^2 K_p T_i s + \omega_n^2 K_p}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 K_p + \omega_n^2 K_p T_i s}{s^3 T_i + 2\delta\omega_n s^2 T_i + \omega_n^2 K_p T_i s + \omega_n^2 K_p}$$

$$\frac{C(s)}{R(s)} = \frac{(K_p/T_i)\omega_n^2 + \omega_n^2 K_p s}{s^3 + 2\delta\omega_n s^2 + \omega_n^2 K_p s + \omega_n^2 (K_p/T_i)}$$

Assume, $K_p/T_i = K_i$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K_i\omega_n^2 + \omega_n^2 K_p s}{s^3 + 2\delta\omega_n s^2 + \omega_n^2 K_p s + \omega_n^2 K_i}}$$

→ It is observed that, the PI - controller introduces zero in the system and increase the order by one.

→ To increase the type number, results in reduce the steady - state error.

PID-Controller:-

The PID-controller produces an output signal consist of three terms,

a) one proportional to error signal

b) proportional to integral of error signal

c) proportional to derivative of error signal

$$u(t) \propto [e(t) + \int e(t)dt + \frac{d}{dt} e(t)]$$

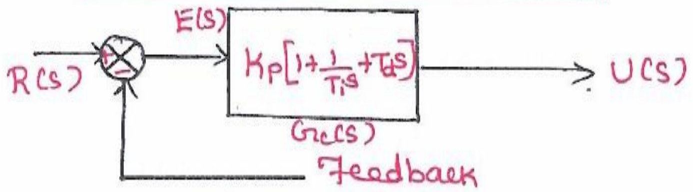
$$u(t) \propto K_p e(t) + \frac{K_p}{T_i} \int e(t)dt + K_p T_d \frac{d}{dt} e(t)$$

Taking Laplace Transform,

$$U(s) = K_p E(s) + \frac{K_p}{T_i s} E(s) + K_p T_d S E(s)$$

$$U(s) = E(s) \cdot K_p [1 + \frac{1}{T_i s} + T_d s]$$

$$\boxed{\frac{U(s)}{E(s)} = K_p [1 + \frac{1}{T_i s} + T_d s]}$$



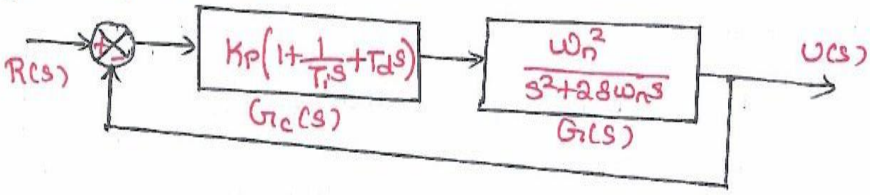
Effect of PID-Controller:-

→ A suitable condition of three basic modes P, I, D can improve all the aspect of the system performance.

→ The P-controller increase the loop gain and stabilize the gain, but produce steady-state error.

→ The I - Controller eliminate the steady state error.

→ The D - controller reduces the rate of change of error.



The new transfer function is,

$$G(s)_{new} = G_c(s) \cdot G(s)$$

$$= K_p\left(1+\frac{1}{T_i s}+T_d s\right) \cdot \frac{\omega_n^2}{s^2+2\delta\omega_n s}$$

$$G(s)_{new} = \frac{K_p\omega_n^2\left(1+\frac{1}{T_i s}+T_d s\right)}{s^2+2\delta\omega_n s}.$$

closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\dfrac{K_p\omega_n^2\left(1+\frac{1}{T_i s}+T_d s\right)}{s^2+2\delta\omega_n s}}{1+\dfrac{K_p\omega_n^2\left(1+\frac{1}{T_i s}+T_d s\right)}{s^2+2\delta\omega_n s}(1)}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K_p\omega_n^2\left(1+\frac{1}{T_i s}+T_d s\right)}{s^2+2\delta\omega_n s+K_p\omega_n^2\left(1+\frac{1}{T_i s}+T_d s\right)}}$$

5) Sketch the root locus of a unity feedback system which has an open loop transfer function,

$$G(s) = \frac{K}{s(s^2+4s+13)}.$$ [N/D-2014].

Solution:-

Step1:- To locate poles and zeros.

The roots of the quadratic equation is,

$$s = \frac{-4 \pm \sqrt{16-(4\times13)}}{2}$$

$$s = -2 \pm j3.$$

$\therefore$ The poles are lying at $s=0, -2+j3$ and $-2-j3$

Step 2:- To find the root locus on real axis.

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis, then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus.

Step 3:- To find angles of asymptotes and centroid.

$$\text{Angle of asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m}, \quad q = 0, 1, \ldots (n-m)$$

Here $n=3$ and $m=0$, $q=0,1,2,3$

When $q=0$, Angles $= \pm\frac{180^\circ}{3} = \pm 60^\circ$

When $q=1$, Angles $= \pm\frac{180\times 3}{3} = \pm 180^\circ$

When $q=2$, Angles $= \pm\frac{180\times 5}{3} = \pm 300^\circ = \mp 60^\circ$

When $q=3$, Angles $= \pm\frac{180\times 7}{3} = \pm 420^\circ = \pm 60^\circ$

$$\text{Centroid} = \frac{\text{Sum of Poles} - \text{Sum of Zeros}}{n-m}$$

$$= \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3}$$

$$\boxed{\text{Centroid} = -1.33}$$

Step-4:- To find the breakaway and breakin points.

Closed loop Transfer Function
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1+\frac{K}{s(s^2+4s+13)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s^3+4s^2+13s+K}$$

The characteristic equation is,

$$s^3+4s^2+13s+K=0$$

$$K = -s^3-4s^2-13s.$$

On differentiating,

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

Put $\frac{dK}{ds} = 0$

$$-(3s^2 + 8s + 13) = 0$$

$$3s^2 + 8s + 13 = 0$$

$$s = \frac{-8 \pm \sqrt{8^2 - 4 \times 13 \times 3}}{2 \times 3}$$

$$\boxed{s = -1.33 \pm j1.6}$$

When, $s = -1.33 + j1.6$,

$$K = -(s^3 + 4s^2 + 13s)$$

$$= -\left[(-1.33 + j1.6)^3 + 4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)\right]$$

$K \neq$ positive and real

When $s = -1.33 - j1.6$,

$K \neq$ not real and positive.

Since the values of K for, $s = -1.33 \pm j1.6$ are not real and positive, these points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step-5: To find the angle of departure.

Draw the vectors from all other poles to the pole P_2. Let the angles of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(3/2) = 123.7°$.

$\theta_2 = 90°$.

Angle of departure $= 180° - (\theta_1 + \theta_2)$

$= 180° - (123.7° + 90°)$

$= -33.7°$

Step-6:- To find the Crossing Point on imaginary axis.

$s^3 + 4s^2 + 13s + K = 0$

Put $s = j\omega$.

$(j\omega)^3 + 4(j\omega)^2 + 13j\omega + K = 0$

$-j\omega^3 - 4\omega^2 + 13j\omega + K = 0$

Equating imaginary part,

$-\omega^3 + 13\omega = 0$

$-\omega(\omega^2 + 13) = 0$

$\omega^2 + 13 = 0$

$\boxed{\omega = \pm 3.6}$

Equating real part,

$-4\omega^2 + K = 0$

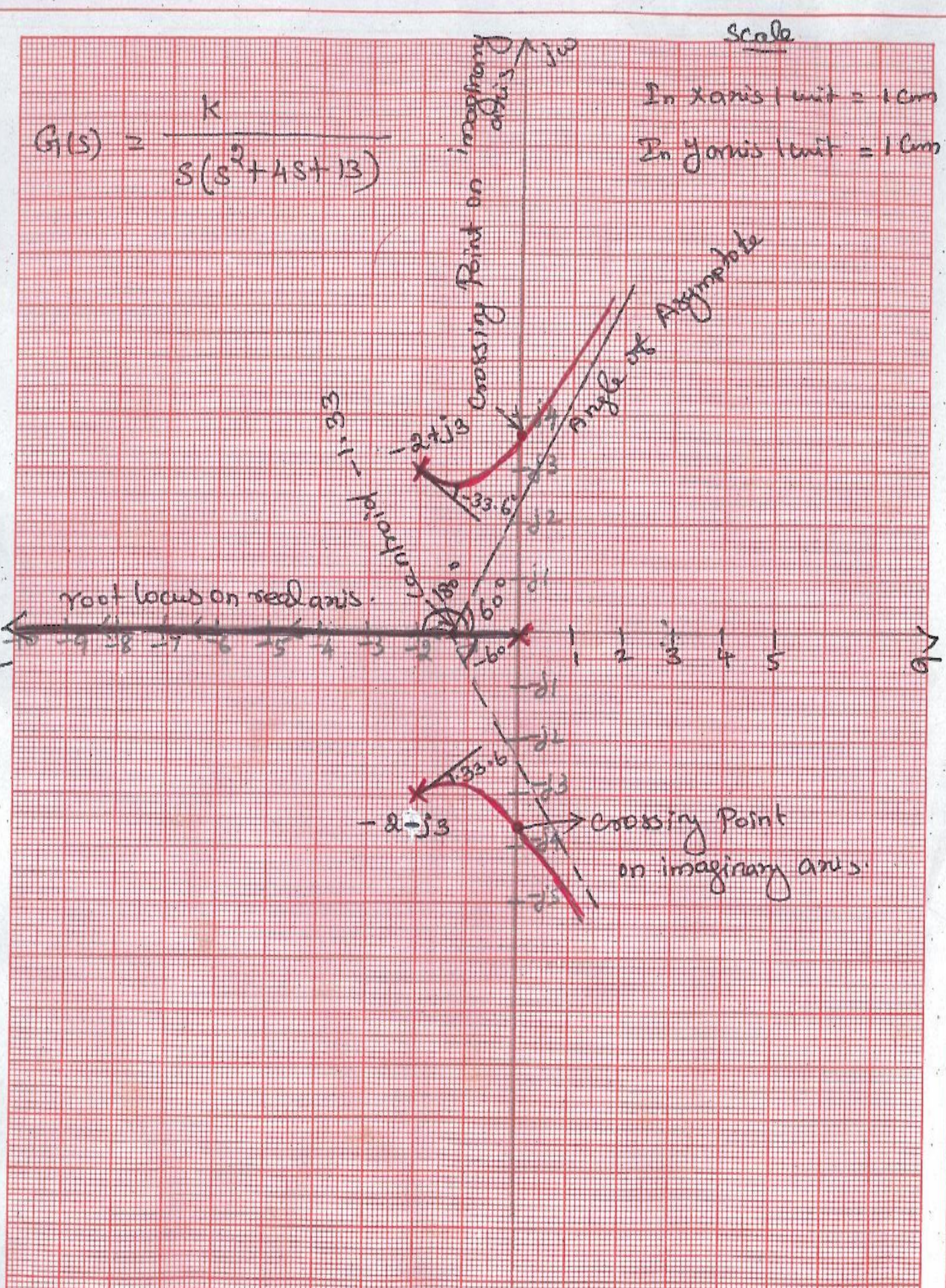
$K = 4\omega^2$

$K = 4(13)$

$\boxed{K = 52}$

The Crossing Point of root locus is $\pm j3.6$

The value of K at this Crossing Point is $K = 52$.



G(s) = K / s(s²+4s+13)
Scale
In x axis 1 unit = 1 cm
In y axis 1 unit = 1 cm
jω
Crossing Point on imaginary axis
Angle of Asymptote
Centroid -1.33
-2+j3
-2-j3
-33.6°
33.6
180°
60°
-60°
root locus on real axis
Crossing Point on imaginary axis
-σ
σ

6) Sketch the root locus of the system whose open loop transfer function is $G(s) = \frac{K}{s(s+2)(s+4)}$. Find the value of K, so that the damping ratio of the closed loop system is 0.5. [N/D-10]

Solution:-

Step-1:- To locate poles and zeros.

The poles are lying at, $s = 0, -2, -4$.

There is no zeros.

Step-2:- To find the root locus on real axis.

* There are three poles on real axis.

* Choose a test point on real axis between $s=-2$ and $s=-4$. To the right of this point, the total number of real poles and zeros is one, which is an odd number. Hence the real axis between $s=0$ and $s=-2$ will be a part of root locus.

* Choose a test point between (-2 to -4), there will not be a part of root locus.

* Choose a test point (-4). To the right of this point, the total number of poles and zeros is 3, which is an odd number. Therefore left of -4 will be part of root locus.

Step-3:- To find asymptotes and centroid.

$$\text{Angle of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}$$

Here $n=3$, $m=0$

When $q=0$, Angles $= \pm \frac{180^\circ}{3} = \pm 60^\circ$

When $q=1$, Angles $= \pm \frac{180^\circ \times 3}{3} = \pm 180^\circ$

$$\text{Centroid} = \frac{\text{Sum of Poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-2-4-0}{3}$$

$$\boxed{\text{Centroid} = -2}$$

Step-4:- To find the breakaway and breakin points.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$

Characteristic equation, $s(s+2)(s+4)+K=0$

$$s^3+6s^2+8s+K=0$$

$$K = -(s^3+6s^2+8s)$$

$$\frac{dK}{ds} = -(3s^2+12s+8)$$

Put $\frac{dK}{ds} = 0$

$$-(3s^2+12s+8)=0$$

$$s = \frac{-12 \pm \sqrt{12^2-4\times3\times8}}{2\times3} = -0.845 \text{ (or) } -3.154$$

When $s = -0.845$,

$$K = -[(-0.845)^3 + 6(-0.845)^2 + 8(-0.845)] = 3.08$$

Since K is positive and real for $s = -0.845$, this point is actual breakaway point.

When $s = -3.154$,

$$K = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)] = -3.08$$

Since K is negative for $s = -3.154$, this is not an actual breakaway point.

Step 5:- To find angle of departure.

Since there are no complex pole or zero, we need not find angle of departure.

Step 6:- To find the crossing point on imaginary axis.

$$s^3 + 6s^2 + 8s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

$$-j\omega^3 - 6\omega^2 + 8j\omega + K = 0$$

Equating imaginary part,

$$-\omega^3 + 8\omega = 0$$

$$-\omega(\omega^2 - 8) = 0$$

$$\omega^2 = 8$$

$$\boxed{\omega = \pm 2.8}$$

Equating real part.

$$-6\omega^2 + K = 0$$

$$K = 6\omega^2$$

$$K = 6(8)$$

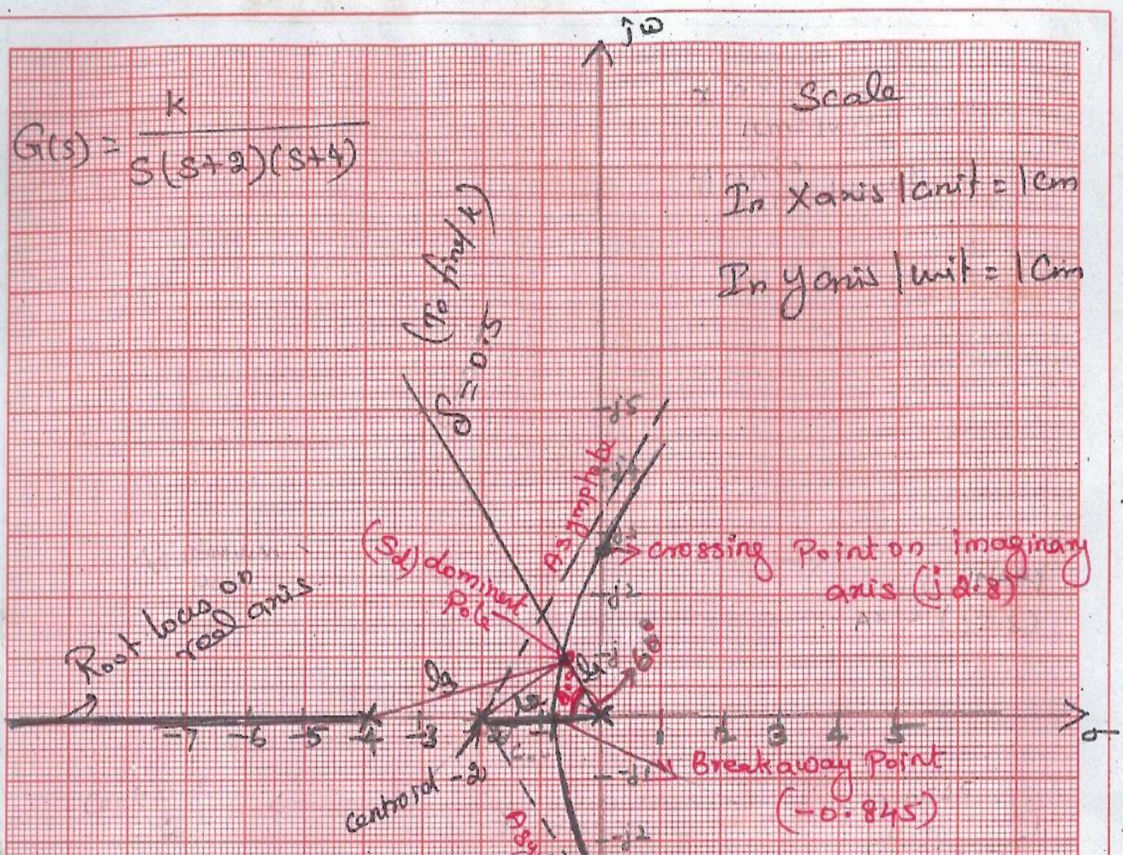
$$\boxed{K = 48}$$

$$G(s) = \frac{k}{S(S+2)(S+4)}$$

Scale

In X axis 1 unit = 1 cm

In Y axis 1 unit = 1 Cm



$l_3 = 3.5$ unit

$l_2 = 1.75$ unit

$l_1 = 1.3$ unit

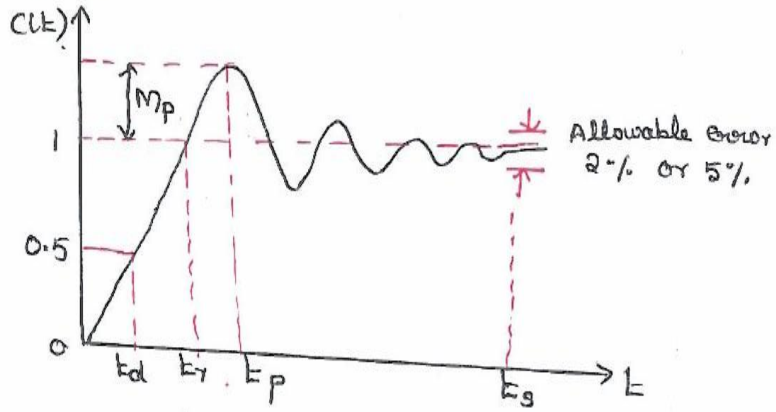
$$K = l_1 l_2 l_3$$

$$= 3.5 \times 1.75 \times 1.3$$

$$\boxed{k = 8}$$

7) Derive the time domain specifications of second order system. [M|J-16]

The desired performance characteristics of control systems are specified interms of time domain specifications.



Delay time:-

It is the time taken for response to reach 50% of the final value, for the very first time.

Rise time:-

It is the time taken for response to raise from 0 to 100% for the very first time.

The unit step response of second order system for underdamped case is given by,

$$C(t) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t+\theta)$$

At $t = t_r$, $C(t) = C(t_r) = 1$.

$$C(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\frac{-e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

Since, $-e^{-\xi\omega_n t_r} \neq 0$,

$$\sin(\omega_d t_r + \theta) = 0$$

$$\omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

Rise time, $\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$

Peak Time:-

It is defined as the time taken for the response to reach the peak value for the very first time.

To find the expression for peak time, differentiate $C(t)$ with respect to t and equate to 0.

$$\left.\frac{d}{dt} C(t)\right|_{t=t_p} = 0$$

Here, $C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

$$\frac{d}{dt} C(t) = -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} (-\xi\omega_n) \sin(\omega_d t + \theta) + \left(\frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\right) \cos(\omega_d t + \theta)\,\omega_d$$

Put $\omega_d = \omega_n \sqrt{1-\varepsilon^2}$.

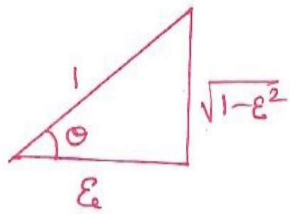
$$\frac{d}{dt} c(t) = \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}} (\varepsilon\omega_n) \sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon\omega_n t} \cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}} \left[\varepsilon \sin(\omega_d t + \theta) - \sqrt{1-\varepsilon^2} \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon\omega_n t} \left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon\omega_n t} \left[\sin(\omega_d t + \theta) - \theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon\omega_n t} \sin(\omega_d t)$$



At $t = t_p$, $\frac{d}{dt} c(t) = 0$

$$\frac{\omega_n}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon\omega_n t_p} \sin(\omega_d t_p) = 0$$

Here, $e^{-\varepsilon\omega_n t_p} \neq 0$. So,

$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = \pi$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

Peak overshoot:-

It is defined as the ratio of the maximum peak value to the final value.

$$\%\text{ Peak overshoot} = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$$

Where, $C(t_p)$ = Peak response at $t = t_p$.

$C(\infty)$ = Final steady state value.

$$C(t) = 1 - \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}} \sin(\omega_d t + \theta)$$

At $t=\infty$, $C(t) = C(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\varepsilon^2}} \sin(\omega_d t + \theta) = 1$

At $t = t_p, C(t) = C(t_p) = 1 - \frac{e^{-\varepsilon\omega_n t_p}}{\sqrt{1-\varepsilon^2}} \sin(\omega_d t_p + \theta)$

$$= 1 - \frac{e^{-\varepsilon\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\varepsilon^2}} \sin(\omega_d t_p + \theta)$$

$$= 1 - \frac{e^{-\varepsilon\omega_n \frac{\pi}{\omega_n\sqrt{1-\varepsilon^2}}} \sin(\pi + \theta)}{\sqrt{1-\varepsilon^2}}$$

$$= 1 + \frac{e^{-\varepsilon\pi/\sqrt{1-\varepsilon^2}}}{\sqrt{1-\varepsilon^2}} \sin\theta.$$

$$\boxed{\%\, M_p = e^{-\frac{\varepsilon\pi}{\sqrt{1-\varepsilon^2}}} \times 100.}$$

Settling time:-

It is defined as the time taken by the response to reach and stay within a specified error.

$t_s = \frac{1}{\varepsilon\omega_n} = 4T$ (for 2% error)

$t_s = \frac{3}{\varepsilon\omega_n} = 3T$ (for 5% error)

8. Draw the root locus of the system where transfer function is given by,

$$G(s) = \frac{K(s+1)}{s(s^2+5s+20)}$$

Sollution:

Step1: Find poles & zeroes

$s=0, -1\pm j3.7, -1-j3.7$ poles

$s=-1$ zeroes

Mark it on graph sheet.

step2: Between 0 & -1 root locus lies on real axis

step3: a) no. of asymptote $= P-Z = 3-1 = 2$

b) Angle of asymptote $= \frac{180(2q+1)}{P-Z}$

$q = 0, 1$

If $q=0$ then $\theta = \frac{180(2(0)+1)}{2} = 90^\circ$

If $q=1$ then $\theta = \frac{180(2(1)+1)}{2} = 270^\circ$

c) Centroid $= \frac{\text{sum of poles} - \text{sum of zeroes}}{P-Z}$

$$= \frac{(-2)-(-1)}{2} = \frac{-2+1}{2} = -0.5$$

Step4: Breakaway point does not exist because it has complex poles

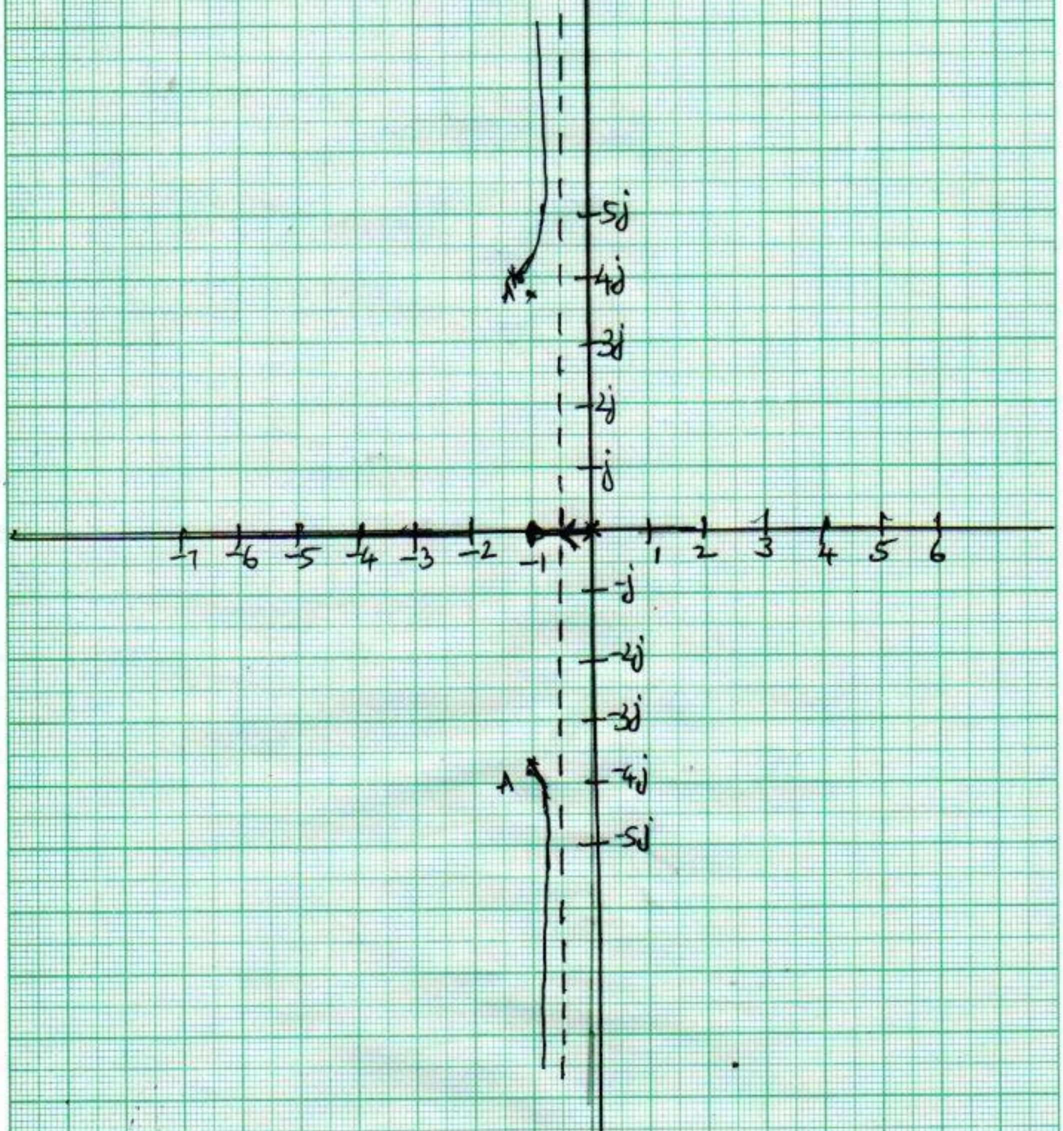
Step5: Angle of departure:

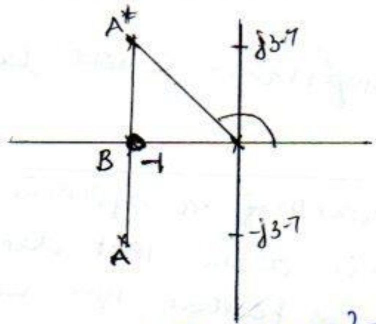
$= 180 -$ (sum of angles subtended by ref. pole with other poles) + sum of angles subtended by ref. pole with other zeroes

$= ~~180~~$

Root locus for

$$G(s) = \frac{K(s+1)}{s(s^2+2s+1)}$$





A^* is ref pole $\{ \angle A^* OB + 180 \}$ = Angle of A^* with $s=0$

$$\angle A^* OB = \tan^{-1}\left(\frac{1 \cdot 37}{1}\right) = 0.9402 \text{ rad} = 53.87^\circ$$

So Angle of A^* with $s=0$

$$180 - 53.87 = 126.13^\circ$$

Angle of A with $A^* = 90$

Angle of B with $A^* = 90^\circ$

So, Angle of departure $= 180 - (126.13 + 90) + 90$

$$= 53.87$$

Step6: Intersection with imaginary axis

characteristic equation:

$$= \frac{K(s+1)}{s(s^2+2s+1)}$$

$$= \frac{K(s+1)}{s^3+2s^2+s+K(s+1)}$$

$$= \frac{K(s+1)}{s^3+2s^2+s(K+1)+K}$$

So characteristic equation is $s^3+2s^2+s(K+1)+K=0$

Using Routh array,

$$\begin{array}{lll} s^3 & 1 & K+1 \\ s^2 & 2 & K \\ s & \frac{+K+2}{2} & 0 \\ s^0 & K & \end{array}$$

$K \geq 0$

$\frac{K+1}{2} = 0$

$K = -1$ It is not proper value. So root locus never touches imaginary axis

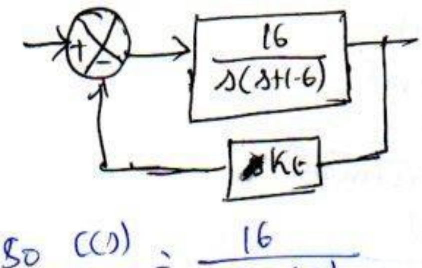
9. The overall transfer function of a system is given by $\frac{C(s)}{R(s)} = \frac{16}{s^2+1.6s+16}$ It is desired that damping ratio is 0.8 Determine Derivative feedback time constant K_t Compare rise time, peak time, overshoot & steady state error for unit ramp function with & without derivative feed back control. [NOV 16]

open loop transfer function $= \frac{16}{s^2+1.6s}$

$= \frac{16}{s(s+1.6)}$

and closed loop TF $= \frac{16}{s(s+1.6)+16} = \frac{16}{s^2+1.6s+16}$ (without derivative feedback)

If derivative feed back is present,



So $\frac{C(s)}{R(s)} = \frac{\frac{16}{s(s+1.6)}}{1+\frac{16\,K_t}{s(s+1.6)}} = \frac{16}{s^2+1.6s+16K_t}$

$= \frac{16}{s^2+1.6s+16K_t}$

If damping ratio is 0.8

$2\xi\omega_n = 1.6$

$2(0.8)\omega_n = 1.6 \Rightarrow \omega_n = 1$

$\omega_n^2 = 16K_t = 1$

$K_t = \frac{1}{16} = 0.0625$

So $\xi = 0.8, \omega_n = 1$

Without derivative feed back $2\xi\omega_n = 1.6$

$\omega_n^2 = 16$ & $\omega_n = 4$

$2\xi(4) = 1.6$

$\xi = \frac{1.6}{8} = 0.2$

Without derivative feed back

$$\text{rise time} = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi - 1.369}{4\sqrt{1-0.2^2}} = \frac{1.772}{3.919} = 0.452 \text{ sec}$$

$$\text{peak time} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14}{3.919} = 0.8012$$

$$\text{overshoot} = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right] = \exp\left[\frac{-0.2\times\pi}{\sqrt{1-0.2^2}}\right]$$

$= 0.526$

With derivative feed back:

$\xi = 0.8, \omega_n = 1$

$$\text{Rise time} \quad t_r = \frac{\pi - \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]}{\omega_n\sqrt{1-\xi^2}} = \frac{2.601}{(1)\sqrt{1-0.8^2}}$$

$= 4.335$ sec [Rise time increases]

peak time,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14}{0.6} = 5.233$$

Peak time increases

$$\text{Overshoot}: \exp\left[\frac{-0.8\pi}{\sqrt{1-0.8^2}}\right] = 0.015$$

overshoot reduces.

10. Derive the response of Undamped & critically damped system with unit step input [Apr 17]

* The standard form of closed loop transfer function of second order system

$$\frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2 + 2\xi\omega n s + \omega n^2}$$

Undamped System

Here $\xi = 0$

$$So, \frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2 + \omega n^2}$$

With $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s}\frac{\omega n^2}{s^2 + \omega n^2} = \frac{A}{s} + \frac{B}{s^2 + \omega n^2}$$

$$A = \cancel{s}\frac{1}{\cancel{s}}\frac{\omega n^2}{s^2 + \omega n^2}\Big|_{s=0} = 1$$

$$B = \cancel{(s^2 + \omega n^2)}\frac{1}{s}\frac{\omega n^2}{\cancel{(s^2 + \omega n^2)}}\Big|_{s = j\omega n} = -j\omega n = -s$$

$$So\ C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega n^2}$$

$$\boxed{c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2 + \omega n^2}\right] = 1 - \cos\omega n t}$$

Critically damped System

Here $\xi = 1$

$$So\ \frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2 + 2\omega n s + \omega n^2} = \frac{\omega n^2}{(s + \omega n)^2}$$

$$R(s) = \frac{1}{s}\quad \&\quad C(s) = \frac{1}{s}\frac{\omega n^2}{(s + \omega n)^2}$$

$$C(s) = \frac{A}{s} + \frac{B}{s + \omega n} + \frac{C}{(s + \omega n)^2}$$

$$A = \cancel{s}\frac{1}{\cancel{s}}\frac{\omega n^2}{(s + \omega n)^2}\Big|_{s=0} = 1$$

$$B = \frac{d}{ds}\left[(s+\omega n)^2 C(s)\right]\bigg|_{s=-\omega n} = \frac{d}{ds}\left(\frac{\omega n^2}{s}\right)\bigg|_{s=-\omega n}$$

$$= \frac{-\omega n^2}{s^2}\bigg|_{s=-\omega n} = -1$$

$$C = (s+\omega n)^2 C(s)\big|_{s=-\omega n} = (s+\omega n)^2 \frac{1}{s}\frac{\omega n^2}{(s+\omega n)^2}\bigg|_{s=-\omega n}$$

$$= -\omega n$$

$$\text{So } C(s) = \frac{1}{s} - \frac{1}{s+\omega n} - \frac{\omega n}{(s+\omega n)^2}$$

$$L^{-1}[C(s)] = c(t) = 1 - e^{-\omega n t} - \omega n t e^{-\omega n t}$$

11. A unity feed back system has an open loop transfer function of $G(s) = \frac{10}{s(s+2)}$. Find rise time, peak time, percentage overshoot and settling time for unit step input of 12 units [Apr 17]

$$CLTF = \frac{10}{s(s+2)+10} = \frac{10}{s^2+2s+10}$$

$$\text{So } s^2 + 2\xi\omega n s + \omega n^2 = s^2 + 2s + 10$$

$$\omega n^2 = 10$$

$$\omega n = 3.162$$

$$2\xi\omega n = 2$$

$$\xi\omega n = 1$$

$$\xi(3.162) = 1$$

$$\xi = 0.316$$

Rise time: $\dfrac{\pi - \tan^{-1}\sqrt{\frac{1-\xi^2}{\xi}}}{\omega n\sqrt{1-\xi^2}} = \dfrac{\pi - 1.249}{(3.162)(0.948)} = \dfrac{1.892}{2.999}$

$= 0.6308$ sec

peak time: $\dfrac{\pi}{\omega n\sqrt{1-\xi^2}} = \dfrac{\pi}{(3.162)(0.948)} = \dfrac{\pi}{1.048} = 2.99$ se overshoot $= M_p = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$

$= \exp\left[\frac{-0.316\times\pi}{\sqrt{1-0.316^2}}\right] = e(-1.046)$

$= \exp\left[-1.046\right] = 0.3513$

$t_s = \frac{4}{\xi\omega_n}$ for 2% $= \frac{4}{0.316\times 3.162} \approx 4$ sec

$= \frac{3}{\xi\omega_n}$ for 5% $= \frac{3}{0.316\times 3.162} = 3$ sec

11. For Servo mechanisms with open loop transfer functions given below, explain what type of input will lead to steady steady state error and calculate their values? [Apr 17]

(1) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$

To get steady state error we give unit ~~step~~ ramp input

$\lim_{s\to 0} sG(s) = K_v$ = velocity error constant

$K_v = \lim_{s\to 0} sG(s) = \cancel{s}\frac{20(s+2)}{\cancel{s}(s+1)(s+3)}$

$= \frac{20(2)}{3} = 13.33$

Steady state error $= \frac{1}{13.33} = 0.075$

(2) $G(s) = \frac{10}{(s+2)(s+3)}$

Here unit step leads to steady state error

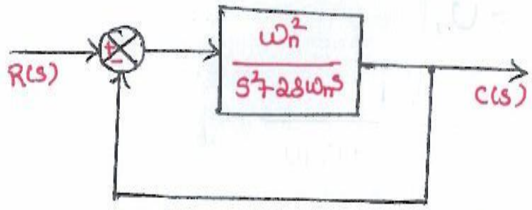
$G(s) = \frac{10}{6} = 1.66 = \lim_{s\to 0} G(s)$

PART-B.

1) Discuss the correlation between time and frequency response of second order system.

Consider a second order system in the unity feedback of an open loop transfer function.

$$G(s) = \frac{\omega_n^2}{s^2+2\delta\omega_n s}$$



The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{\omega_n^2}{s^2+2\delta\omega_n s}}{1+\frac{\omega_n^2}{s^2+2\delta\omega_n s}(1)}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

where,

$\omega_n \rightarrow$ natural frequency

$\delta \rightarrow$ damping ratio.

Put $s=j\omega$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(j\omega)^2+2\delta\omega_n(j\omega)+\omega_n^2}$$

divide by ω_n^2 both numerator and denominator,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2/\omega_n^2}{-\frac{\omega^2}{\omega_n^2}+\frac{j2\delta\omega_n\omega}{\omega_n^2}+\frac{\omega_n^2}{\omega_n^2}}$$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{-\frac{\omega^2}{\omega_n^2}+\frac{j2\delta\omega}{\omega_n}+1}$$

Assume $\frac{\omega}{\omega_n} = U$,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{1-U^2+j2\delta U}$$

magnitude, $$|M(j\omega)| = \frac{1}{\sqrt{(1-U^2)^2+(2\delta U)^2}}$$

$$|M(j\omega)| = \frac{1}{\sqrt{1-2U^2+U^4+4\delta^2U^2}}$$

Phase angle, $\angle M(j\omega) = -\tan^{-1}\left(\frac{2\delta U}{1-U^2}\right)$

The resonance frequency can be obtained by differentiating the magnitude with respect to "U" and equating to "zero"

Therefore, $\frac{dM}{dU} = 0$

$$M = [1-2U^2+U^4+4\delta^2U^2]^{1/2}$$

$$\frac{dm}{dU} = [0-4U+4U^3+8U\delta^2]\cdot -\frac{1}{2}[1-2U^2+U^4+4\delta^2U^2]^{-3/2}$$

$$0 = \frac{-\frac{1}{2}(-4U+4U^3+8\delta^2U)}{(1-2U^2+U^4+4\delta^2U^2)^{3/2}}$$

$$0 = 4u^3 - 4u + 8\delta^2 u$$

$$0 = (4u^2 - 4 + 8\delta^2)u$$

$$4u^2 - 4 + 8\delta^2 = 0$$

$$4u^2 = 4 - 8\delta^2$$

$$u^2 = \frac{4 - 8\delta^2}{4} = 1 - 2\delta^2$$

$$u = \sqrt{1 - 2\delta^2}$$

we know that,

$$u = \frac{\omega}{\omega_n}$$

Resonance frequency,

$$u_r = \frac{\omega_r}{\omega_n}$$

$$\frac{\omega_r}{\omega_n} = \sqrt{1 - 2\delta^2}$$

$$\omega_r = \omega_n \sqrt{1 - 2\delta^2}$$

Substitute the value of u_r in magnitude and phase angle equation,

$$M(j\omega) = \frac{1}{\sqrt{1 - 2(1-2\delta^2) + (1-2\delta^2)^2 + 4\delta^2(1-2\delta^2)}}$$

$$M(j\omega) = \frac{1}{\sqrt{1 - 2 + 4\delta^2 + 1 - 4\delta^2 + 4\delta^4 + 4\delta^2 - 8\delta^4}}$$

$$M(j\omega) = \frac{1}{\sqrt{4\delta^2 - 4\delta^4}} = \frac{1}{\sqrt{4\delta^2(1-\delta^2)}}$$

$$\boxed{M(j\omega) = \frac{1}{2\delta\sqrt{1-\delta^2}}}$$

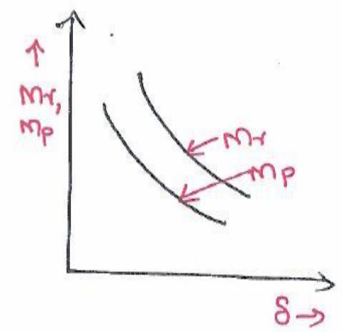
$$\alpha_r = -\tan^{-1}\left(\frac{2\delta u}{1-u^2}\right) = -\tan^{-1}\left(\frac{2\delta\sqrt{1-2\delta^2}}{1 - 1 + 2\delta^2}\right)$$

$$= -\tan^{-1}\left(\frac{2\delta\sqrt{1-\delta^2}}{2\delta^2}\right)$$

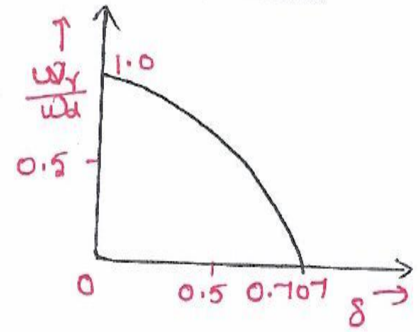
$$\boxed{\alpha_r = -\tan^{-1}\left(\frac{\sqrt{1-\delta^2}}{\delta}\right)}$$

This is the expression for the correlation between time domain and frequency domain specification.

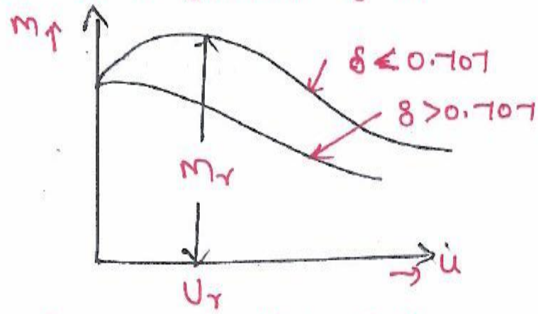
M_r Vs δ



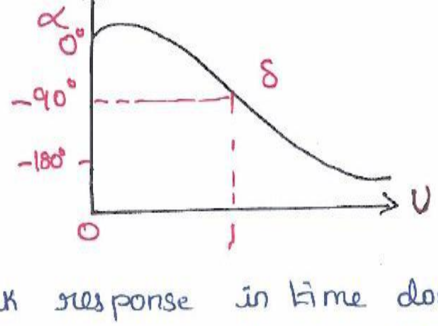
ω_r/ω_d Vs δ



Magnitude as a function of U,



Phase angle as a function of υ,



Peak response in time domain and frequency response in time domain depends only on damping factor δ.

$$M_r = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

$$\alpha_r = -\tan^{-1}\left(\frac{\sqrt{1-\delta^2}}{\delta}\right)$$

$$M_p = e^{-\delta\pi/\sqrt{1-\delta^2}}$$

2) Sketch the Bode plot and determine the system gain K for gain cross-over frequency $\omega_{gc} = 5\,rad/sec$,

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$ [N/D-2010] [M/J-2013]

Solution:-

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in $G(s)$,

$$G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Let $K=1$, $$G(j\omega) = \frac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Magnitude Plot.

Corner frequencies are,

$$\omega_{c1} = \frac{1}{0.2} = 5\ rad/sec$$

$$\omega_{c2} = \frac{1}{0.02} = 50\ rad/sec.$$

Term	Corner Frequency (rad/s)	Slope (db/dec)	Change in slope (db/dec)
$(j\omega)^2$	—	+40	—
$\frac{1}{1+j0.2\omega}$	5	-20	40-20=20
$\frac{1}{1+j0.02\omega}$	50	-20	-20+20=0

Choose, $\omega_l = 0.5$ rad/sec

$\omega_h = 100$ rad/sec

Let us calulate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h.

At $\omega = \omega_l$, $A = 20\log|(j\omega)^2| = 20\log(\omega)^2 = 20\log(0.5)^2$

$\boxed{A = -12db}$

At $\omega = \omega_{c1}$, $A = 20\log|(j\omega)^2| = 20\log(5)^2$

$\boxed{A = 28db}$

At $\omega = \omega_{c2}$, $A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\omega=\omega_{c1})}$

$= 20 \times \log\frac{50}{5} + 28$

$\boxed{A = 48db}$

At $\omega = \omega_h$, $A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\omega=\omega_{c2})}$

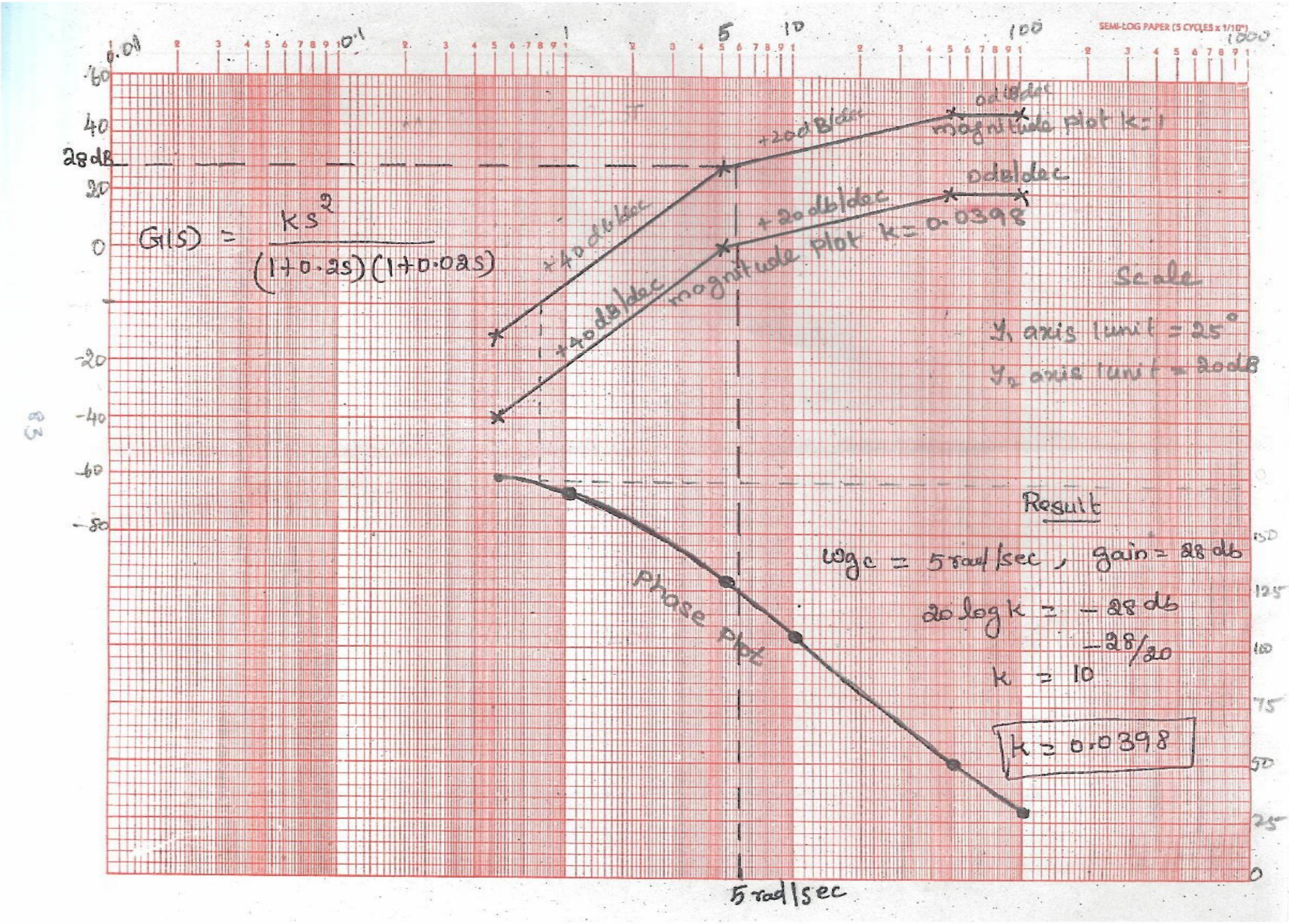
$= 0 \times \log\frac{100}{50} + 48$

$\boxed{A = 48\,db.}$

Phase plot:-

$\phi = \angle G(j\omega) = 180° - \tan^{-1}0.2\omega - \tan^{-1}0.02\omega.$

ω(rad/sec)	$\tan^{-1}0.2\omega$	$\tan^{-1}0.02\omega$	ϕ
0.5	5.7	0.6	173.7 = 174
1	11.3	1.1	167.6 ≃ 168
5	45	5.7	129.3 ≃ 130
10	63.4	11.3	105.3 ≃ 106
50	84.3	45	50.7 ≃ 50
100	87.1	63.4	29.5 ≃ 30


SEMI-LOG PAPER (5 CYCLES x 1/10")
0.01
0.1
1
5
10
100
1000
$G(s) = \frac{ks^2}{(1+0.2s)(1+0.02s)}$
28dB
+40 db/dec
+20dB/dec
0dB/dec
magnitude plot k=1
+40dB/dec
+20db/dec
0dB/dec
magnitude plot k=0.0398
Phase plot
5 rad/sec
Scale
y_1 axis 1unit = 25°
y_2 axis 1unit = 20dB
Result
ω_{gc} = 5 rad/sec, gain = 28 db
20 log k = −28 db
k = $10^{-28/20}$
k = 0.0398

Calculation of K:-

The gain cross-over frequency (ω_{gc}) is 5 rad/sec. The gain is 28 db. If gain cross-over frequency is 5 rad/sec, then at that frequency, the db gain should be zero.

At ω_{gc} = 5 rad/sec, Gain = 28

$$20 \log K = -28 \text{ db.}$$

$$\log K = -28/20$$

$$K = ln^{(-28/20)}$$

$$\boxed{K = 0.0398}$$

Hence, $G(s) = \dfrac{0.0398 s^2}{(1+0.2s)(1+0.02s)}$

3) Draw the Bode plot and obtain gain margin and phase margin. Given transfer function, $G(s) = \dfrac{75(1+0.2s)}{s(s^2+16s+100)}$

[M/J-2014]

Solution:-

$$s^2+16s+100 = s^2+2\xi\omega_n s+\omega_n^2$$

On comparing, we get

$\omega_n^2 = 100$; $\omega_n = 10$

$2\xi\omega_n = 16$; $\xi = \dfrac{16}{2\omega_n} = \dfrac{16}{2\times 10} = 0.8$

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} = \frac{75(1+0.2s)}{s\times100\left(1+\frac{s^2}{100}+\frac{16s}{100}\right)}$$

$$G(s) = \frac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$.

$$G(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega(1+0.01(j\omega)^2+0.16j\omega)}$$

$$G(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega(1-0.01\omega^2+j0.16\omega)}$$

Magnitude plot.

corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec

$\omega_{c2} = \omega_n = 10$ rad/sec

Choose, $\omega_l = 0.5$ rad/sec

$\omega_h = 20$ rad/sec.

Term	corner frequency rad/sec	Slope db/dec	change in slope db/dec
$\frac{0.75}{j\omega}$	—	-20	—
$1+0.2j\omega$	5	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	10	-40	$0-40=-40$

At $\omega = \omega_l$, $A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{0.5}$

$$\boxed{A = 3.5\,db}$$

At $\omega = \omega_{c1}$, $A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\left|\frac{0.75}{5}\right|$

$$\boxed{A = -16.5\,db}$$

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\omega=\omega_{c1})}$

$$= 0 \times \log\left(\frac{10}{5}\right) + (-16.5)$$

$$\boxed{A = -16.5\,db}$$

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\omega=\omega_2)}$

$$= -40 \times \log\frac{20}{10} + (-16.5)$$

$$\boxed{A = -28.5\,db}$$

Phase plot.

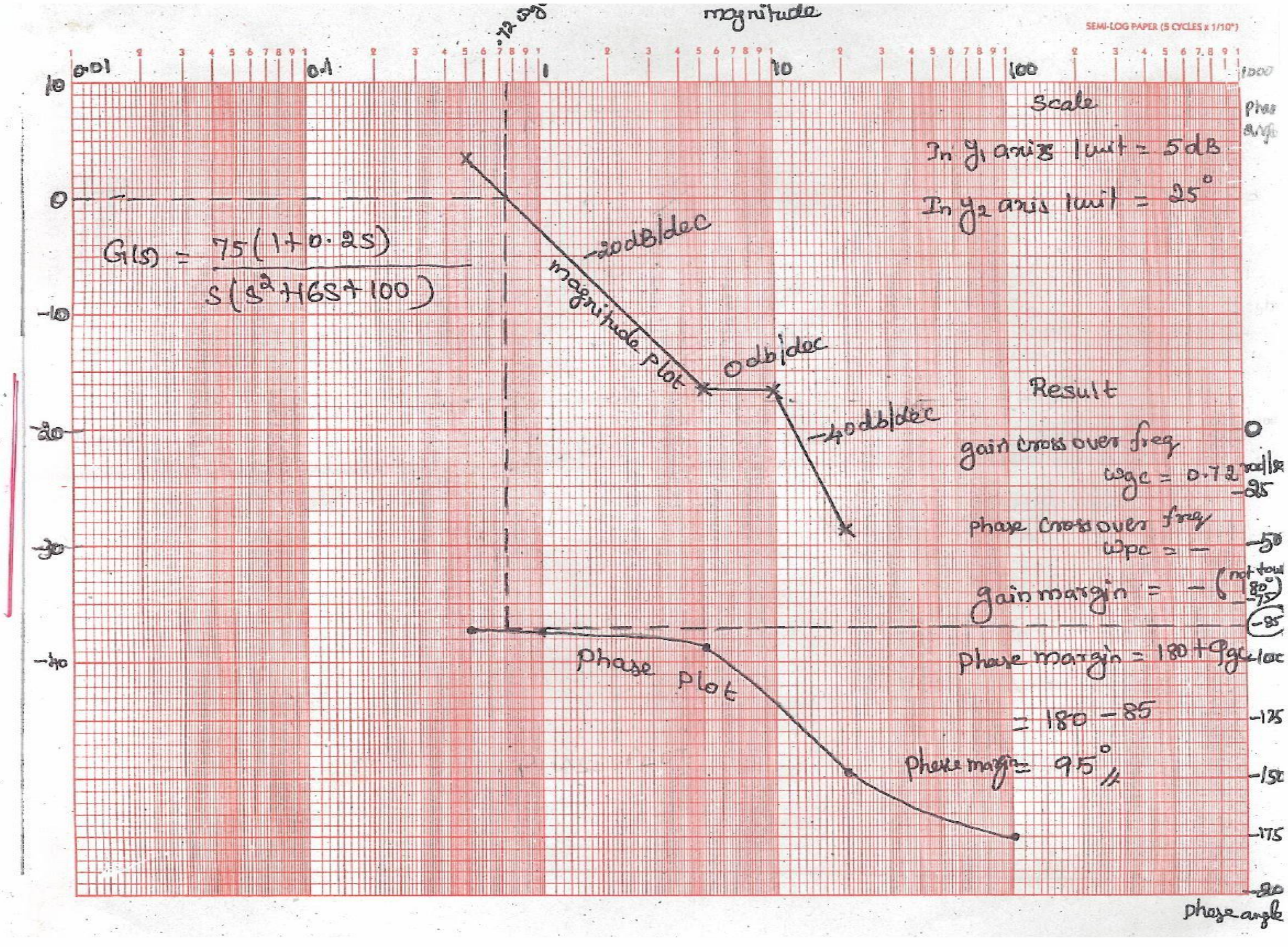
$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} \quad \text{for } \omega \le \omega_h$$

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left[\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right] \quad \text{for } \omega > \omega_h$$

ω rad/sec	$\tan^{-1}0.2\omega$ deg	$\tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$ deg	$\phi = \angle G(j\omega)$
0.5	6.7	4.6	$-88.9 \simeq -88$
1	11.3	9.2	$-87.9 \simeq -88$
5	45	46.8	$-91.8 \simeq -92$
10	63.4	90	$-116.6 \simeq -116$
20	75.9	$-46.8+180 = 133.2$	$-147.3 \simeq -148$
50	84.3	$-18.4+180 = 161.6$	$-167.3 \simeq -168$
100	87.1	$-92+180 = 170.8$	$-173.7 \simeq -174$

Result :-

Phase Margin $= 92^\circ$, Gain margin $= \infty$


magnitude
SEMI-LOG PAPER (5 CYCLES x 1/10")
0.72 rad/s
0.01
0.1
1
10
100
1000
10
0
-10
-20
-30
-40
G(s) = 75(1+0.25) / s(s²+16s+100)
-20dB/dec
magnitude plot
0db/dec
-40db/dec
Phase plot
Scale
In y1 axis 1unit = 5dB
In y2 axis 1unit = 25°
Result
gain cross over freq
ωgc = 0.72 rad/sec
phase cross over freq
ωpc = —
gain margin = —
Phase margin = 180 + φgc
= 180 − 85
phase margin = 95°
phase angle
0
-25
-50
-75
-80°
-85
-100
-125
-150
-175
-200

4) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s^2(1+s)(1+2s)}$. Sketch the polar plot and determine the gain margin and phase margin.

[N|D-10, A|M-11, N|D-15]

Solution:-

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

Put $s = j\omega$,

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

Corner frequencies, $\omega_{c1} = 1/2 = 0.5$ rad/sec.

$\omega_{c2} = 1/1 = 1$ rad/sec.

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)\ (1+j2\omega)} = \frac{1}{\omega\angle 90^\circ \sqrt{1+\omega^2}\angle \tan^{-1}\omega \sqrt{1+4\omega^2}\angle \tan^{-1}2\omega}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

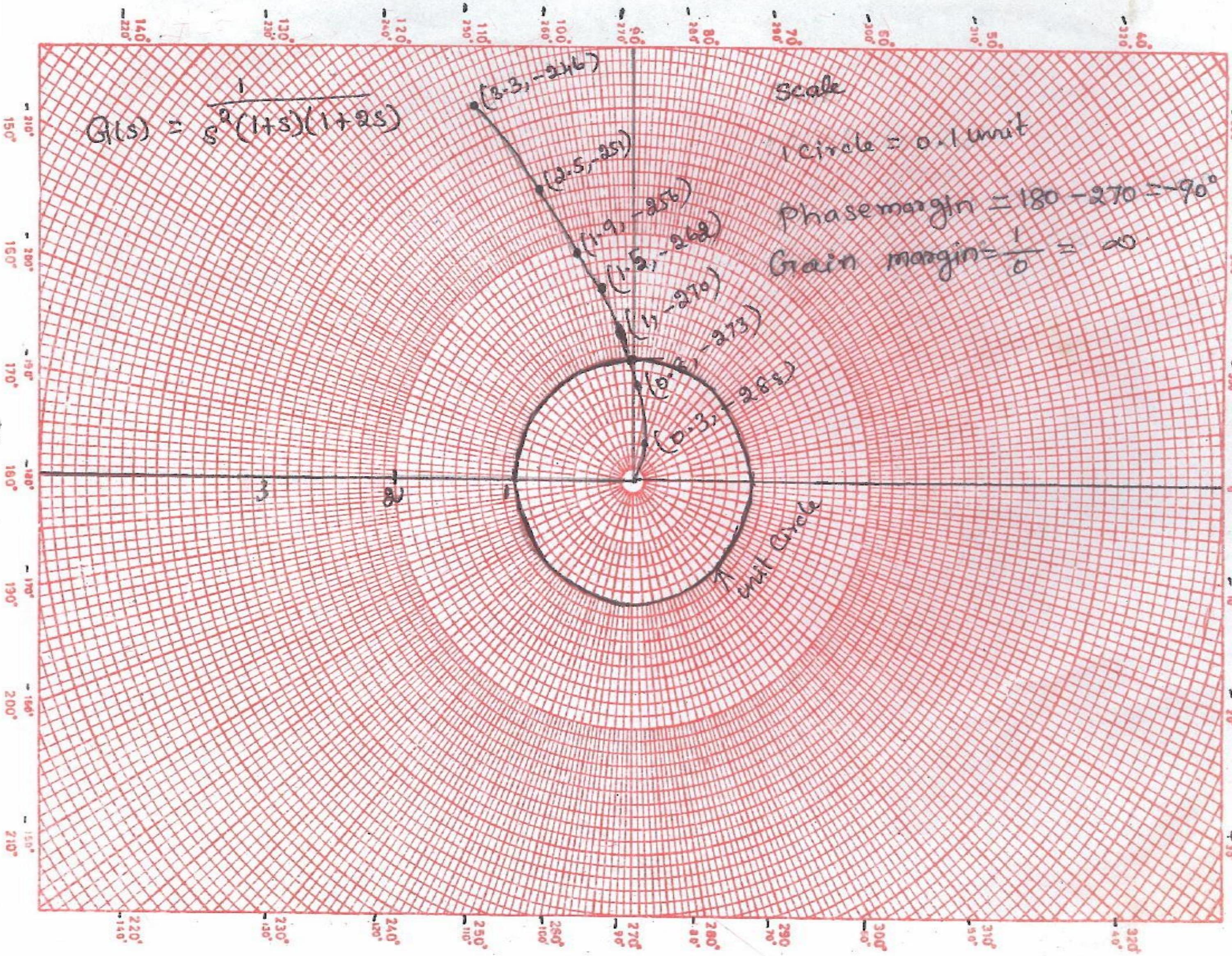
ω	0.35	0.4	0.45	0.5	0.6	0.7	0.8	0.9	1	2
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.92	0.68	0.5	0.4	0.31	0.08
$\angle G(j\omega)$	-144	-153	-156	-161	-171	-179	-186	-189	-198	-229

Gain margin, $K_g = \frac{1}{|G(j\omega)|} = \frac{1}{0.7}$.

$$\boxed{K_g = 1.4286}$$

Phase margin, $\gamma = 180 + \phi_{gc} = 180 - 168$

$$\boxed{\gamma = 12^\circ}$$


$G(s) = \frac{1}{s^2(1+s)(1+2s)}$
Scale
1 circle = 0.1 unit
Phase margin = 180 − 270 = −90°
Gain margin = $\frac{1}{0}$ = ∞
(3.3, −246)
(2.5, −251)
(1.9, −256)
(1.5, −262)
(1, −270)
Unit Circle
1
2
3

6. The open loop transfer function of a system is given by $G(s) = \frac{1}{s(s+1)(2s+1)}$. Sketch the polar plot & determine gain margin & phase margin? [NOV16]

Solution

Magnitude

put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}2\omega$$

ω	Magnitude	phase angle
0.1	$(0.1024)^{-1} = 9.76$	-107.02
0.2	$(0.2196)^{-1} = 4.553$	-123.11
0.5	$(0.7905)^{-1} = 1.265$	-161.5
1	$(3.162)^{-1} = 0.3162$	-198.43
2	$(18.43)^{-1} = 0.054$	-229.3
5	$(256.22)^{-1} = 0.0039$	-252.9
~~10~~		

7. Draw the Bode plot of Transfer function $G(s) = \frac{1}{s(s^2+3s+5)}$ Determine gain margin & phase margin? [NOV16]

Step1: Convert into time constant form

$$G(s) = \frac{1/5}{s[1+0.6s+0.2s^2]} = \frac{1/5}{s[1+0.6s+0.2s^2]}$$

Step2: Find CF $\omega_n^2 = 5$, $\omega_n = 2.2306$

Magnitude plot Step3: Prepare the table

6. The open loop transfer function of a system is given by $G(s) = \frac{1}{s(s+1)(2s+1)}$. Sketch the polar plot & determine gain margin & phase margin? [NOV16]

Solution

Magnitude

put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}2\omega$$

ω	Magnitude	phase angle
0.1	$(0.1024)^{-1} = 9.76$	-107.02
0.2	$(0.2196)^{-1} = 4.553$	-123.11
0.5	$(0.7905)^{-1} = 1.265$	-161.5
1	$(3.162)^{-1} = 0.3162$	-198.43
2	$(18.43)^{-1} = 0.054$	-229.3
5	$(256.22)^{-1} = 0.0039$	-252.9
~~10~~		

7. Draw the Bode plot of Transfer function $G(s) = \frac{1}{s(s^2+3s+5)}$. Determine gain margin & phase margin? [NOV 16]

Step1: Convert into time constant form

$$G(s) = \frac{1/5}{s[1+0.6s+0.2s^2]} = \frac{1/5}{s[1+0.6s+0.2s^2]}$$

Step2: Find cf $\omega_n^2 = 5$, $\omega_n = 2.2306$

Magnitude plot Step3: Prepare the table

Term	CF	Change in slope	New slope
$\frac{1}{j\omega}$	—	-20	—
$\frac{1}{s^2+3s+5}$	$2.2306 = \omega_n$	-40	-60

Let $\omega_l = 0.1$ rad/sec

gain at $\omega = 0.1$

$$= 20\log\frac{1}{0.1} = 20\,dB$$

gain at $\omega = 2.2306$

$$= 20\log\frac{1}{2.2306} = -6.96$$

choosing $\omega_h = 10$ rad/sec

$$= -6.96 - 60\log\left[\frac{10}{2.2306}\right] = -46.05$$

phase plot

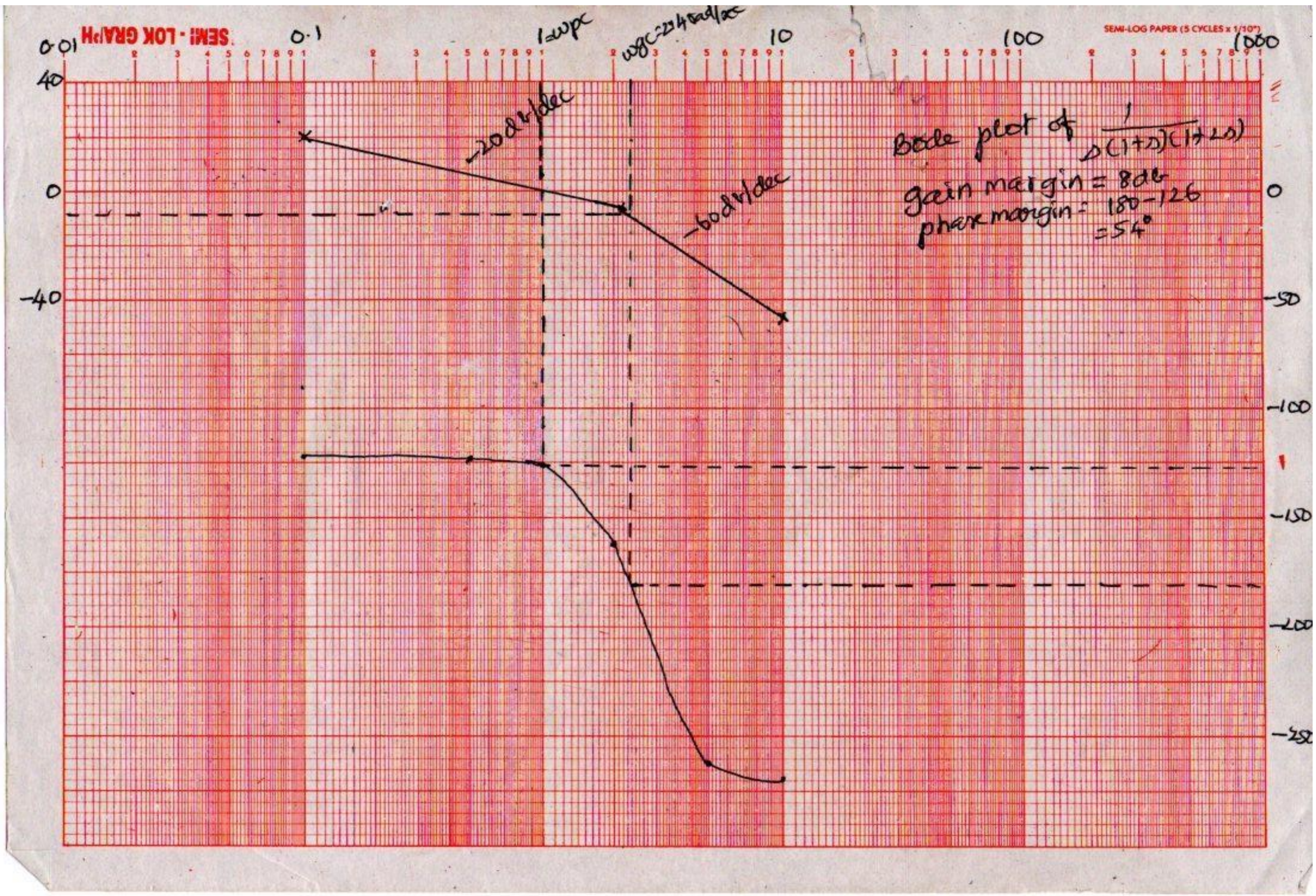
Step 4: Find phase angle expression

put $s = j\omega$

$$G(s) = \frac{0.2}{j\omega(1 - 0.2\omega^2 + 0.6j\omega)}$$

$$\angle G(j\omega) = -90 - \tan^{-1}\left[\frac{0.6}{1 - 0.2\omega^2}\right]$$

ω	phase angle $-90-\tan^{-1}\left(\frac{0.6}{1-0.2\omega^2}\right)$	$\frac{0.6}{1-0.2\omega^2}$	$\tan^{-1}\left(\frac{0.6}{1-0.2\omega^2}\right)$	Total angle
0.1	-121	0.601	31	-121
0.5	-122.3	0.634	32.3	-122.3
1	-126.86	0.75	36.86	-126.86
2	-161.5	3	71.5	-161.5
5	$-261.$	-0.15	$-8.5+180 = 171$	-261
10	-268	-0.03	$-1.71+180 = 178$	-268



Bode plot of $\frac{1}{s(1+s)(1+2s)}$
gain margin = 8dB
phase margin = 180-126
=54°
-20dB/dec
-60dB/dec
1=ωpc
ωgc=2.4rad/sec
SEMI - LOK GRAPH
SEMI-LOG PAPER (5 CYCLES x 1/10")
0.01
0.1
10
100
1000
40
0
-40
-50
-100
-150
-200
-250

8. Sketch the Bode plot and determine gain and phase margin for the open loop transfer function of a unity feedback system is given by,

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)} \quad [Apr\ 17]$$

Step 1:

$s = j\omega$

$$G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}$$

Step 2:

Corner frequencies are

$|1+0.4j\omega| = 0$

$0.4\omega = 1$

$\omega = \frac{1}{0.4} = 2.5\ rad/sec$

$|1+0.1j\omega| = 0$

$0.1\omega = 1$

$\omega = \frac{1}{0.1} = 10\ rad/sec.$

Magnitude plot

Prepare a Table of following format

Term	CF	Change in slope	New slope
$\frac{10}{j\omega}$	—	-20	—
$\frac{1}{1+0.4j\omega}$	2.5 rad/sec	-20	-40
$\frac{1}{1+0.1j\omega}$	10 rad/sec	-20	-60

choose $\omega_L = 0.1$ rad/sec,

Gain at $\omega_L = 0.1$

$= 20\log\frac{10}{\omega} = 20\log\frac{10}{0.1} = 40\ db$

gain at $\omega = 2.5$ rad/sec

$= 20\log\frac{10}{2.5} = 12.04\ db$

gain at $\omega = 10$ rad/sec

$= 12.04 - 40\log\left(\frac{10}{2.5}\right) = -12.04\ db$

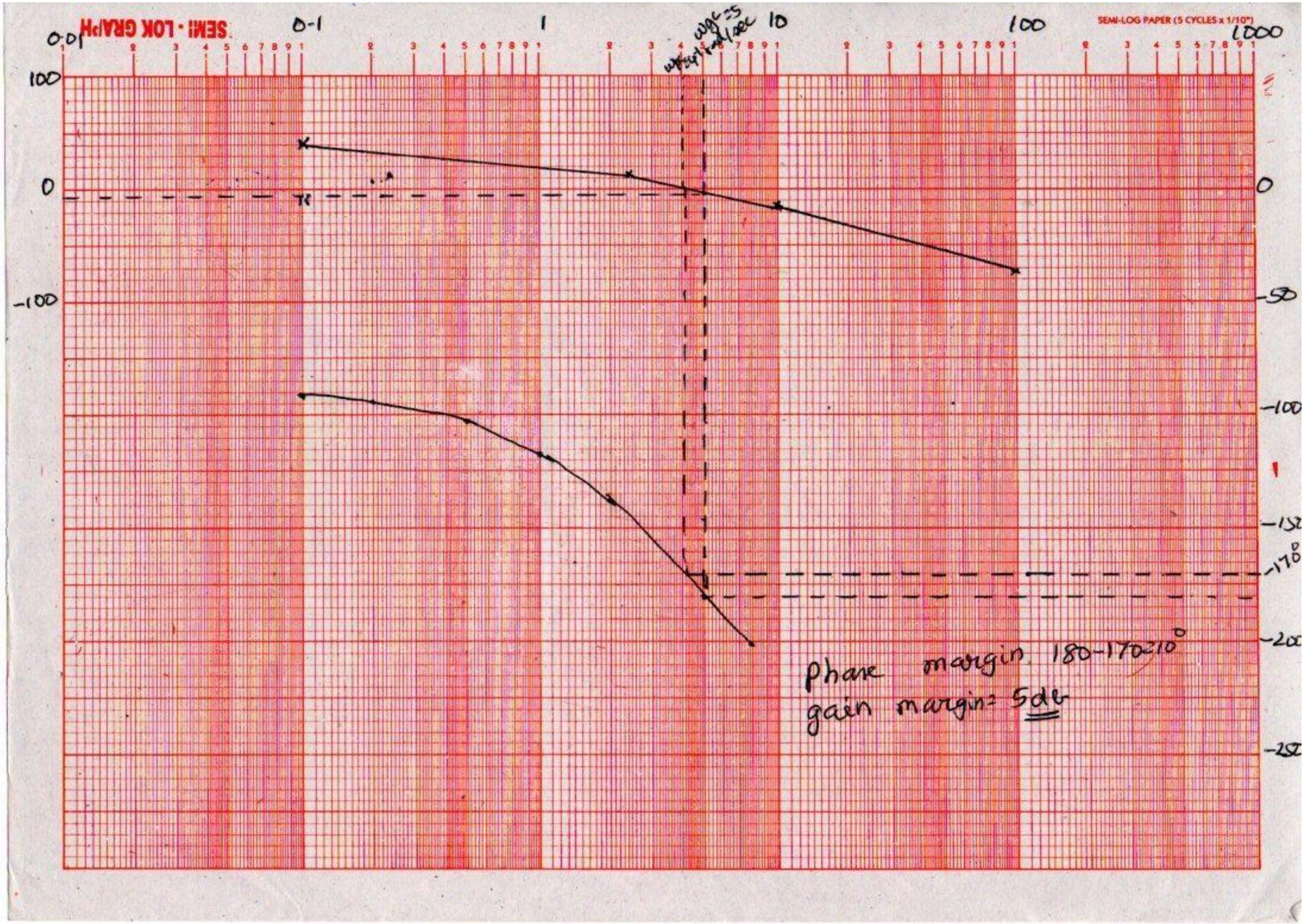
gain at $\omega = 100$ rad/sec

$= -12.04 - 60\log\left(\frac{100}{10}\right) = -72.04\ db$

Phase plot

$= -90 - \tan^{-1}(0.4\omega) - \tan^{-1}(0.1\omega)$

ω	phase angle
0.1	-92.8
0.2	-95.7
0.5	-104.172
1	-117.51
2	-139.9
5	-180
10	-201.3


SEMI-LOG PAPER (5 CYCLES x 1/10")
SEMI - LOK GRAPH
0.01
0.1
1
10
100
1000
100
0
-100
0
-50
-100
-150
-170°
-200
-250
ωgc=5
phase margin 180-170=10°
gain margin= 5db

PART-B.

1)a) Construct the Routh array and determine the stability of the system whose characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. [N/D-14, M/J-13]

Solution:-

The characteristic equation of the system is,

$$s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0.$$

s^6 : 1 8 20 16

s^5 : 2 12 16

s^5 : 1 6 8

s^4 : 1 6 8

s^3 : 0 0

s^3 : 1 3

s^2 : 3 8

s^1 : 0.33

s^0 : 8

s^4 : $\frac{1\times8-6\times1}{1}$ $\frac{1\times20-8\times1}{1}$ $\frac{1\times16-0\times1}{1}$

s^4 : 2 12 16

s^4 : 1 6 8

s^3 : $\frac{1\times6-6\times1}{1}$ $\frac{1\times8-8\times1}{1}$

s^3 : 0 0

The auxiliary equation is,

$A=s^4+6s^2+8$.

$\frac{dA}{ds}=4s^3+12s$

s^3 : 4 12

s^3 : 1 3

s^2 : $\frac{1\times6-3\times1}{1}$ $\frac{1\times8-0\times1}{1}$

s^2 : 3 8

s^1 : $\frac{3\times3-8\times1}{3}$

s^1 : 0.33

s^0 : $0.33\times8-0\times3/0.33$

s^0 : 8

On examining the elements of 1st column of routh array, there is no sign change. Hence the system is limitedly or marginally stable.

Auxiliary polynomial, $s^4+6s^2+8=0$

$x^2+6x+8=0$

roots are, $x=-2$ or -4.

The roots of auxiliary polynomial is, $s=\pm\sqrt{-2}$ & $\pm\sqrt{-4}$

Roots are $+j\sqrt{2}, -j\sqrt{2}, +j2, -j2$.

Result:-

System is limitedly or marginally stable.

b) Determine the stability of the system corresponding to the characteristic equation $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$. Comment on the location of roots using routh array. [M/J-2014]

Solution:-

The characteristic equation is $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

$s^5 : 1 \quad 2 \quad 3$

$s^4 : 1 \quad 2 \quad 5$

$s^3 : \dfrac{1\times2-2\times1}{1} \quad \dfrac{1\times3-5\times1}{1}$

$s^3 : 0 \quad -2$

Replace 0 by ε

$s^3 : \varepsilon \quad -2$

$s^2 : \dfrac{\varepsilon\times2-(-2\times1)}{\varepsilon} \quad \dfrac{\varepsilon\times5-0\times1}{\varepsilon}$

$s^2 : \dfrac{2\varepsilon+2}{\varepsilon} \quad 5$

$s^1 : \dfrac{\dfrac{2\varepsilon+2}{\varepsilon}\times(-2)-(5\times\varepsilon)}{\dfrac{2\varepsilon+2}{\varepsilon}}$

$s^1 : \dfrac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}$

$s^0 : \dfrac{\dfrac{(-5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}\times5-0\times\dfrac{2\varepsilon+2}{\varepsilon}}{-\dfrac{(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}}$

$s^0 : 5$

$s^5 : 1 \quad 2 \quad 3$

$s^4 : 1 \quad 2 \quad 5$

$s^3 : \varepsilon \quad -2$

$s^2 : \dfrac{2\varepsilon+2}{\varepsilon} \quad 5$

$s^1 : \dfrac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}$

$s^0 : 5$

on letting $\varepsilon \to 0$, we get:

$s^5 : 1 \quad 2 \quad 3$

$s^4 : 1 \quad 2 \quad 5$

$s^3 : 0 \quad -2$

$s^2 : \infty \quad 5$

$s^1 : -2$

$s^0 : 5$

Result:-

* There are two sign changes in first column of routh array. Hence the system is unstable.

* Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

2) The OLTF of a unity feedback system is given by $G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$. By applying Routh Criterion discuss the stability of closed loop system as a function of K. Also determine the value of K which will cause sustained oscillation in the closed loop system. Determine the oscillating frequencies. [N/D-2013]

Solution:-

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\frac{K}{(s+2)(s+4)(s^2+6s+25)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

The characteristic equation is,

$(s+2)(s+4)(s^2+6s+25)+K=0 \Rightarrow (s^2+6s+8)(s^2+6s+25)+K=0$

$s^4+12s^3+69s^2+198s+200+K=0.$

s^4:	1	69
s^3:	12	198
s^3:	1	16.5
s^2:	52.5	200+K
s^1:	$\frac{666.25-K}{52.5}$	
s^0:	200+K	

$s^2: \frac{1\times 69-16.5\times 1}{1} \qquad \frac{1\times(200+K)}{1}$

$s^2: 52.5 \qquad 200+K$

$s^1: \frac{52.5\times 16.5-(200+K)\times 1}{52.5}$

$s^1: \frac{666.25-K}{52.5}$

$s^0: \frac{\frac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$

$s^0: 200+K$

For the system to be stable, there should not be any sign change. Hence choose the value of K so that the first column elements are positive.

From s^1 row, the system to be stable, $(666.25 - K) > 0$.

Since $(666.25 - K) > 0$, should be less than 666.25.

From s^0 row, for the system to be stable, $(200 + K) > 0$

Since $(200 + K) > 0$, K should be greater than -200, but practical values of K starts from 0. Hence K should be greater than zero.

$\therefore$ The range of K is $0 < K < 666.25$.

When K = 666.25, the auxiliary equation becomes,

$$52.5s^2 + 200 + K = 0$$

$$s^2 = \frac{-200 - 666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5} = \pm j4.06.$$

$$\boxed{\omega = 4.06 \text{ rad/sec.}}$$

Result:-

* The range of K for stability is $0 < K < 666.25$
* The system oscillates when $K = 666.25$
* The frequency of oscillation, $\omega = 4.06$ rad/sec.

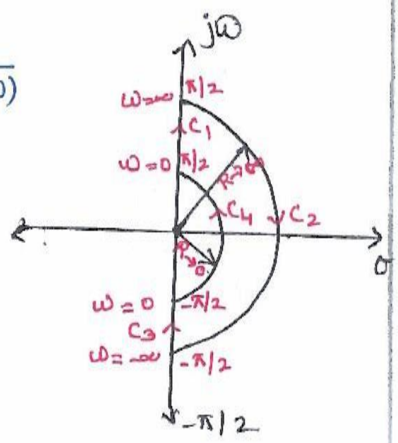
3) Draw the Nyquist plot for the system whose open loop transfer function $G(s)H(s) = \frac{K}{s(s+2)(s+10)}$. Determine the range of K for which closed loop system is stable. [May-11]

Solution:-

$$G(s)H(s) = \frac{K}{s(s+2)(s+10)}$$

$$G(s)H(s) = \frac{K}{20s(1+0.5s)(1+0.1s)}$$

Nyquist contour



$$G(j\omega)H(j\omega) = \frac{0.05K}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$= \frac{0.05K}{j\omega(1+j0.6\omega - 0.05\omega^2)}$$

$$= \frac{0.05K}{(1-0.05\omega^2)j\omega - 0.6\omega^2}$$

$$G(j\omega)H(j\omega) = \frac{0.05K}{-0.6\omega^2 + j\omega(1-0.05\omega^2)}$$

Imaginary Part = 0

$$\omega(1-0.05\,\omega_{pc}^2) = 0$$

$$0 = -0.05\,\omega_{pc}^2 + 1$$

$$\omega_{pc}^2 = \frac{1}{0.05}$$

$$\boxed{\omega_{pc} = 4.47\ \text{rad/sec}}$$

$$G(j\omega)H(j\omega) = \frac{0.05K}{-0.6(4.47)}$$

$$\boxed{G(j\omega)H(j\omega) = -0.00417\,K}$$

Mapping of contour C_1: [$\omega = 0$ to $\omega = +\infty$]

$$G(s)H(s) = \frac{0.05K}{s(1+0.5s)(1+0.1s)}$$

$$s = j\omega$$

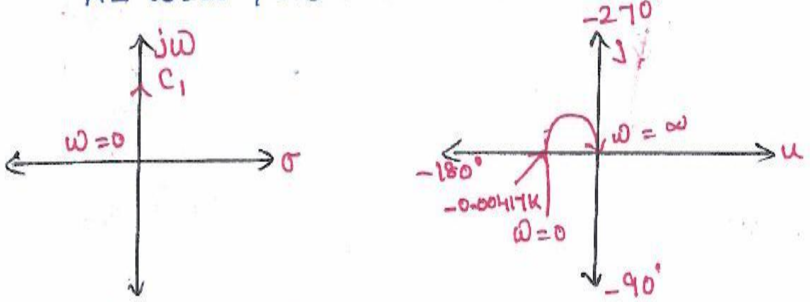
$$G(j\omega)H(j\omega) = \frac{0.05K}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$= \frac{0.05K}{\omega\angle 90^\circ \cdot \sqrt{1+(0.5\omega)^2}\,\angle \tan^{-1}0.5\omega \cdot \sqrt{1+(0.1\omega)^2}\,\angle \tan^{-1}0.1\omega}$$

$$G(j\omega)H(j\omega) = \frac{0.05K}{\omega\sqrt{1+(0.5\omega)^2}\sqrt{(1+(0.1\omega)^2)}}\angle -90 - \tan^{-1}0.5\omega - \tan^{-1}0.1\omega$$

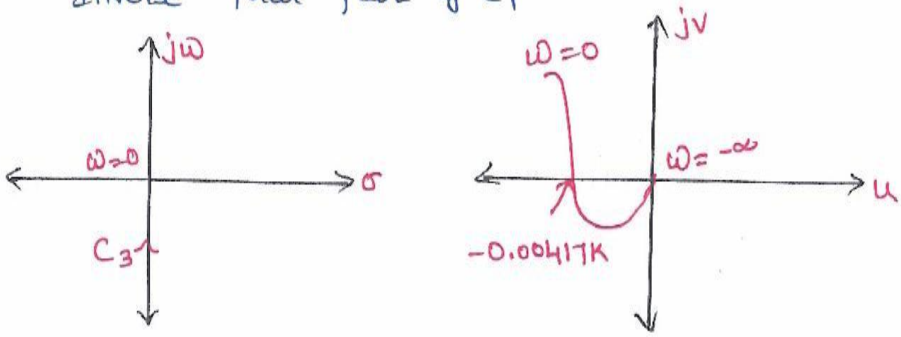
At $\omega = 0$, $G(j\omega)H(j\omega) = \infty \angle -90^\circ$

At $\omega = \infty$, $G(j\omega)H(j\omega) = 0 \angle -270^\circ$.



Mapping of contour C_3: [$\omega = -\infty$ to $\omega = 0$]

Inverse Polar plot of C_1



Mapping of contour C_2: $[\theta = \pi/2 \text{ to } -\pi/2]$

For $R \to \infty \Rightarrow 1+ST = ST$ & $s = \lim_{R\to\infty} Re^{j\theta}$

$$G(s)H(s) = \frac{0.05K}{s(1+0.5s)(1+0.1s)}$$

$$= \frac{0.05K}{s(0.5s)(0.1s)}$$

$$= \frac{0.05K}{s(0.05s^2)} = \frac{K}{s^3}$$

$$\boxed{G(s)H(s) = \frac{K}{s^3}}$$

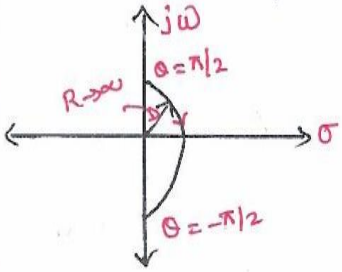
Substitute, $s = \lim_{R\to\infty} Re^{j\theta}$.

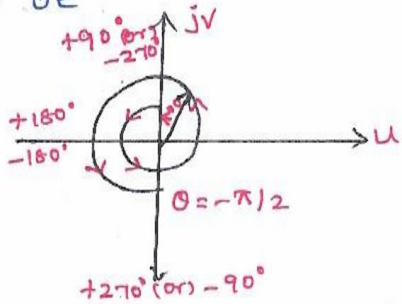
$$G(s)H(s) = \frac{K}{\lim_{R\to\infty}(Re^{j\theta})^3} = \frac{K}{\infty(e^{j\theta})^3}$$

$$G(s)H(s) = 0e^{-j3\theta}$$

When $\theta = \pi/2$, $G(s)H(s) = 0e^{-j3\pi/2}$

When $\theta = -\pi/2$, $G(s)H(s) = 0e^{j3\pi/2}$.





Mapping of contour C_4: $[\theta = -\pi/2 \text{ to } \pi/2]$

For $R \to 0 \Rightarrow 1+ST = 1$ & $s = \lim_{R\to 0} Re^{j\theta}$

$$G(s)H(s) = \frac{0.05K}{s(1+0.5s)(1+0.1s)} = \frac{0.05K}{s(1)(1)}$$

$$\boxed{G(s)H(s) = \frac{0.05K}{s}}$$

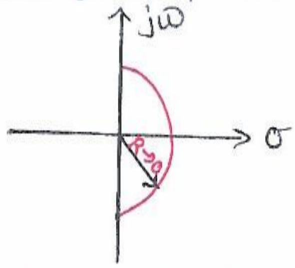
Sub, $s = \lim_{R \to 0} Re^{j\theta}$

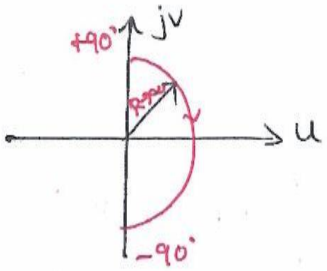
$$G(s)H(s) = \frac{0.05K}{\lim_{R \to 0} Re^{j\theta}} = \frac{0.05K}{0\, e^{j\theta}}$$

$$\boxed{G(s)H(s) = \infty e^{-j\theta}}$$

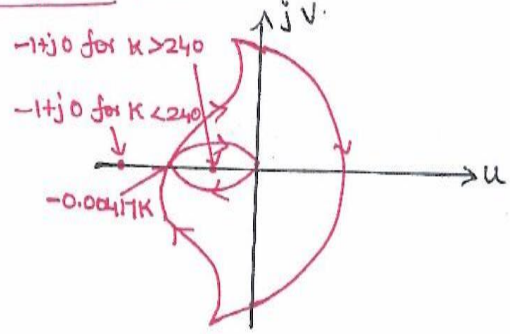
When $\theta = -\pi/2$, $G(s)H(s) = \infty e^{j\pi/2}$

When $\theta = \pi/2$, $G(s)H(s) = \infty e^{-j\pi/2}$





Complete Nyquist Plot.



Limiting value of K: $(-1 + j0)$ Point

$$-0.00417K = -1$$

$$\boxed{K = 240}$$

$K < 240$ $\left[\begin{array}{l} -0.00417K \Rightarrow K = 230 \Rightarrow -0.96 \\ -0.00417K \Rightarrow K = 220 \Rightarrow -0.92 \end{array}\right]$ lies less than -1.

* No encirclement.

$K > 240$ $\left[\begin{array}{l} -0.00417K \Rightarrow K = 250 \Rightarrow -1.04 \\ -0.00417K \Rightarrow K = 260 \Rightarrow -1.08 \end{array}\right]$ lies greater than -1.

* There are 2 encirclement in clockwise direction.

* Gives no poles on right half of s-plane.

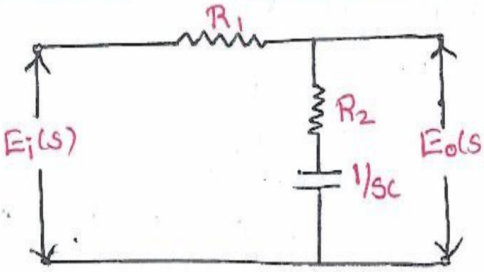
* Two clockwise encircle.

4) Draw the electrical equivalent circuit of lag Compensator and obtain its transfer function. Also explain the design procedure of lag compensator. [M|J-16]

Lag Compensator:-

Compensator having the characteristics of lag network is called lag compensator. The lag compensator is a low pass filter which increases the order of system by 1.

Transfer Function of Lag Compensator:-



$$E_o(s) = \frac{E_i(s) \times \left[R_2 + \frac{1}{sC}\right]}{R_1 + R_2 + \frac{1}{sC}}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + \frac{1}{sC}}{R_1 + R_2 + 1/sC}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{(R_2 sC + 1)/\cancel{sC}}{(R_1 sC + R_2 sC + 1)/\cancel{sC}}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 sC + 1}{R_1 sC + R_2 sC + 1}$$

$$\frac{E_o(S)}{E_i(S)} = \frac{R_2 \cancel{C} \left(S + 1/R_2C \right)}{(R_1+R_2)\cancel{C}\left[S+\frac{1}{R_1R_2C}\right]}$$

multiply & divided by R_2,

$$\frac{E_o(S)}{E_i(S)} = \frac{R_2}{R_1+R_2} \; \frac{\left(S + \frac{1}{R_2C}\right)}{\left[S + \frac{1}{((R_1+R_2)/R_2)\,R_2C}\right]} \rightarrow (1)$$

But the transfer function of lag compensator is given by,

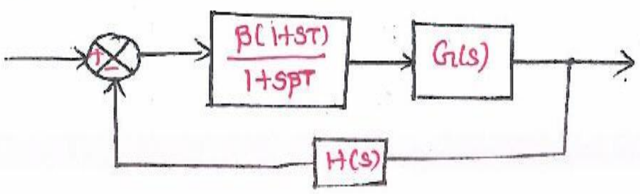
$$G_c(S) = \frac{S+\frac{1}{T}}{S+\frac{1}{\beta T}} \rightarrow (2)$$

On comparing (1) & (2),

$$\frac{E_o(S)}{E_i(S)} = \frac{1}{\beta}\;\frac{\left(S+\frac{1}{T}\right)}{\left(S+\frac{1}{\beta T}\right)}$$

where, $T = R_2C$ and $\beta = (R_1+R_2)/R_2$.

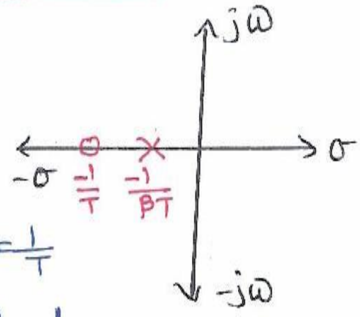
The transfer function of RC network is similar to the general form with an attenuation of $1/\beta$. If the attenuation is not required, then an amplifier with gain β can be connected in cascade with RC network to nullify the attenuation.



S-Plane representation of lag Compensator.

$$G_c(s) = \frac{s+1/T}{s+1/\beta T}$$

where, $T>0$ and $\beta>1$,



Zero of lag compensator, $Z_c = \frac{1}{T}$

Pole of lag compensator, $P_c = \frac{1}{\beta T}$

Design Procedure of lag compensator.

The following steps may be followed to design a lag compensator using bode plot and to be connected in series with transfer function of uncompensated system.

Step-1: Choose the value of K in uncompensated system to meet the steady state error requirement.

Step-2: Sketch the bode plot of uncompensated system.

Step-3: Determine the phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

Step-4: Choose a suitable value for the phase margin of the compensated system.

Let $\gamma_d \rightarrow$ desired phase margin

$\gamma_n \rightarrow$ phase margin of compensated system.

Now, $\gamma_n = \gamma_d + \epsilon$.

Where, $\epsilon \rightarrow$ Additional phase lag to compensate for shift in gain cross-over frequency.

Choose an initial value of $\epsilon = 5°$.

Step-5:- Determine the new gain cross-over frequency, ω_{gcn}. The new ω_{gcn} is the frequency corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let, ϕ_{gcn} = phase of $G(j\omega)$ at new gain cross-over frequency, ω_{gcn}.

Now, $\gamma = 180 + \phi$ (or) $\phi_{gcn} = \gamma_n - 180°$.

Step 6:- Determine the parameter, β of the compensator. The value of β is given by the magnitude of $G(j\omega)$ at new gain cross-over frequency, ω_{gcn}. Find the db gain (A_{gcn}) at new gain cross-over frequency, ω_{gcn}

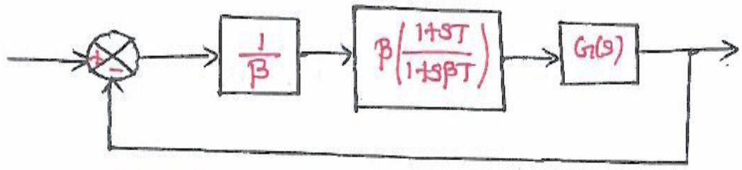
$$A_{gcn} = 20\log\beta \text{ (or) } \beta = 10^{A_{gcn}/20}.$$

Step-7:- Determine the transfer function of lag compensator.

Place the zero of the compensator arbitrarily at 1/10th of the new gain crossover frequency, ω_{gcn}. $Z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$

$$T = \frac{10}{\omega_{gcn}}$$

$$G_c(s) = \frac{s+1/T}{s+1/\beta T} = \beta\left[\frac{1+sT}{1+s\beta T}\right]$$



Step-8:- Determine the open loop transfer function of compensated system. The lag compensator is connected in series with the plant.

If the gain product is not satisfied, then the attenuator with gain ($1/\beta$) can be added in series with the lag compensator.

The open loop transfer function of the compensated system is,

$$G_o(s) = \frac{1}{\beta} G_c(s) \cdot G(s)$$

$$= \frac{1}{\beta} \cdot \beta \frac{(1+sT)}{(1+s\beta T)} G(s)$$

$$\boxed{G_o(s) = \left(\frac{1+sT}{1+s\beta T}\right) G(s)}$$

Step-9:- Determine the actual phase margin of compensated system.

$$\gamma_o = 180^\circ + \phi_{gco}$$

If the actual phase margin is satisfied, then the design is accepted otherwise repeat the procedure from step 4 to 9 by taking $\epsilon = 5^\circ$ more than previous design.

5) Draw the electrical equivalent circuit of lead compensator and obtain its transfer function. Also explain the design procedure of lead compensator. [ND-15]

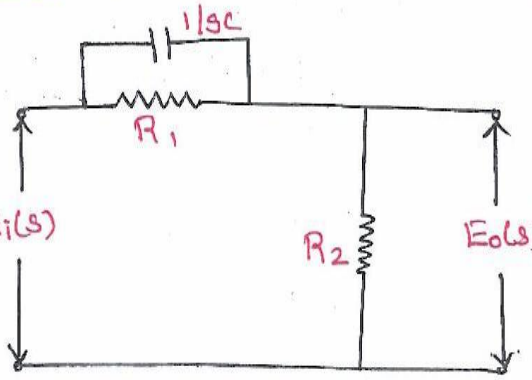
Lead compensator:-

→ A compensator having characteristics of lead network is called lead compensator.

→ If a sinusoidal input applied to a lead network, the steady state output will have phase lead with respect to the input.

→ The lead compensator increase the bandwidth, increases the speed of response and also improve the transient response.

→ Lead compensator is a high pass filter.



$$E_o(s) = E_i(s) \times \frac{R_2}{R_2 + (R_1 \| 1/sC)}$$

$$E_o(s) = E_i(s)\ \frac{R_2}{R_2 + \left(\frac{R_1 \times 1/sC}{R_1 + \frac{1}{sC}}\right)}$$

$$= E_i(s) \cdot \frac{R_2}{R_2 + \left[\frac{R_1 / \cancel{s}C}{\frac{R_1 sC + 1}{\cancel{sC}}}\right]}$$

$$E_o(s) = E_i(s) \cdot \frac{R_2}{R_2 + \frac{R_1}{R_1 sC + 1}}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{\frac{R_2(1 + R_1 sC) + R_1}{R_1 sC + 1}} = \frac{R_2(R_1 sC + 1)}{R_2 R_1 sC + R_2 + R_1}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 \cancel{R_2} C}{R_1 \cancel{R_2} C}\left[\frac{s + 1/R_1 C}{s + \frac{R_1 + R_2}{R_1 R_2 C}}\right]$$

$$\frac{E_o(s)}{E_i(s)} = \frac{s + 1/R_1 C}{s + \frac{1}{R_1 C}\left(\frac{1}{R_2 / R_1 + R_2}\right)}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{s + 1/T}{s + 1/\alpha T}}$$

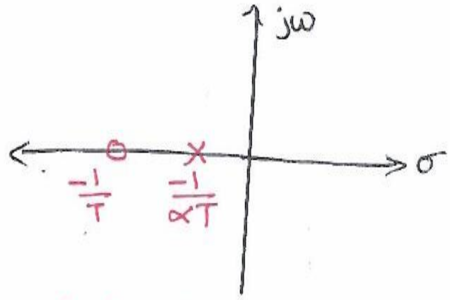
Where,

$$T = R_1 C, \quad \alpha = \frac{R_2}{R_1 + R_2}$$

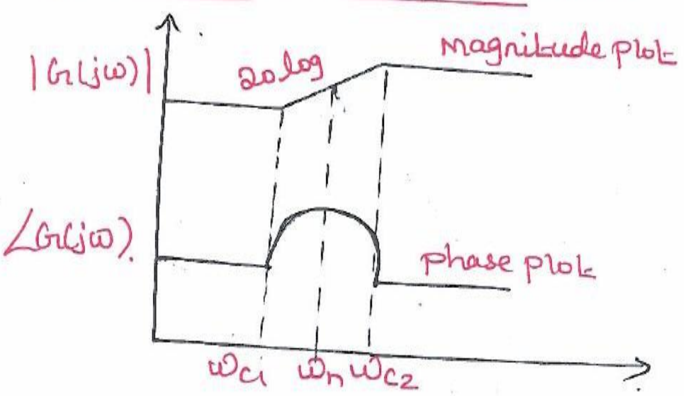
S-plane representation of lead compensator:-

Pole of lead compensator, $P_c = -\frac{1}{\alpha T}$

Zero of lead compensator, $Z_c = -\frac{1}{T}$

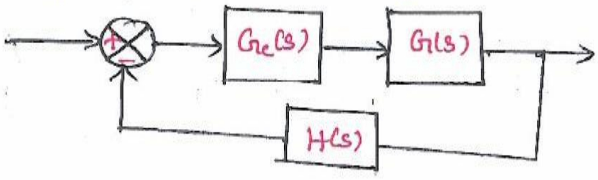


Bode plot of lead compensator:-



Procedure for design of lead compensator using bode plot:-

The following steps may be followed to design the lead compensator using bode plot and to be connected in series with transfer function of compensated system.



Step-1:- Choose the value of K in uncompensated system to meet the steady state error requirement.

Step-2:- Draw the bode plot of uncompensated system.

Step-3:- Determine the phase margin of uncompensated system from bode plot. If the phase margin does not satisfy the requirement then the lead compensator is added.

Step-4:- Determine the amount of phase angle to be contributed by the lead network using the formula is given below.

$$\phi_m = \gamma_d - \gamma + \varepsilon.$$

Where,

$\phi_m \rightarrow$ phase lead angle.

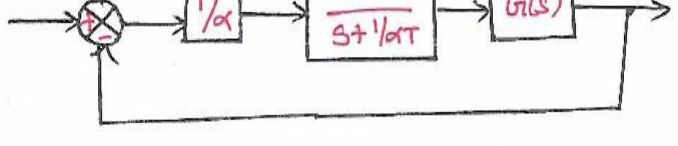
Step-5:- Determine the transfer function of lead compensator.

$$G_c(s) = \frac{s+1/T}{s+1/\alpha T}$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

$$T = \frac{1}{\omega_m\sqrt{\alpha}}$$

Determine the open loop transfer function of compensated system.



The open loop transfer function of overall system,

$$G_o(s) = \frac{1}{\alpha} \cdot \frac{s+1/T}{s+1/\alpha T} \times G(s)$$

$$= \frac{1}{\alpha} \cdot \frac{\alpha(1+sT)}{(1+s\alpha T)} G(s)$$

$$\boxed{G_o(s) = \frac{1+sT}{1+s\alpha T} G(s)}$$

Step-6:- Verify the design. Finally the bode plot of the compensated system is drawn and verify whether it satisfies the specification

If the phase margin of the compensated system is less than the required phase margin, then repeat the step 4 to 6, taking ϵ as 5° more than the previous design.

6 (a) Determine the range of values of K for which the system described by the following characteristic equation is stable? [NOV 16]

s^3	1	K+2	
s^2	3K	4	$\frac{3K(K+2)-4}{3K}$
s	$\frac{3K^2+6K-4}{3K}$	0	$=\frac{3K^2+6K-4}{3K}$
s^0	4		

Now $\frac{3K^2+6K-4}{3K}=0$

$3K^2+6K-4=0$

$K=0.5266, -2.526$

But K has only +ve values

So $K=0.5266$

If $K<0.5266$, Then $3K^2+6K-4=+ve$ otherwise

-ve. So $0<K<0.5266$

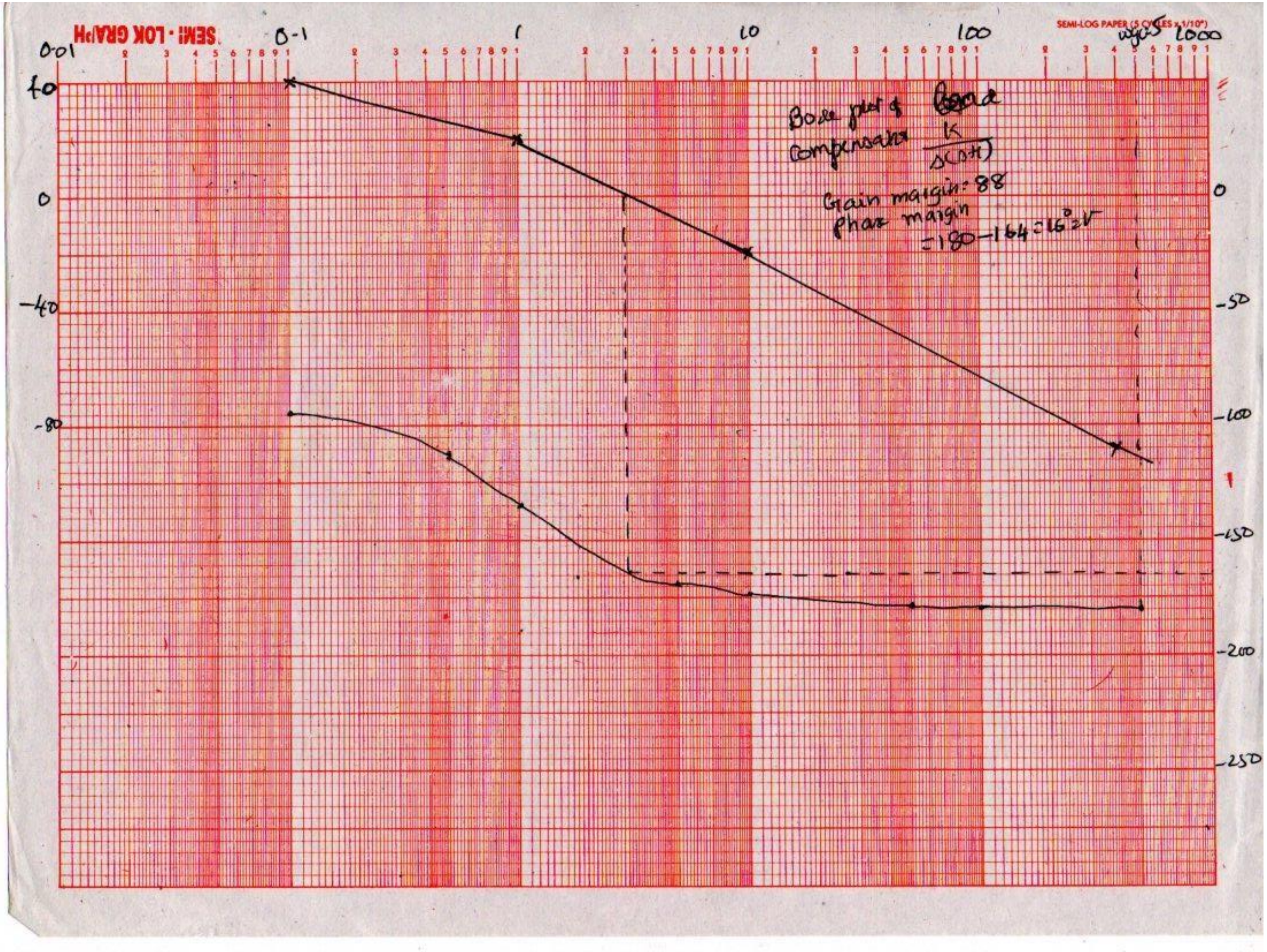
7. Design a lead compensator for a unity feedback system with open loop transfer function $G(s)=\frac{K}{s(s+1)}$ for the specifications $K_v=10\,sec^{-1}$ and phase margin $\phi_M=35^0$? [NOV 16]

Step1: Find the value of $K_v=\lim_{s\to 0}\frac{\cancel{s}\,K}{\cancel{s}(s+1)}=10$

$K=10$

So OLTF $=\frac{10}{s+1}$

Step2: Draw the Bode plot



Bode plot of uncompensated
$\frac{K}{s(s+1)}$
Gain margin = 88
Phase margin
= 180 − 164 = 16°
SEMI-LOK GRAPH
SEMI-LOG PAPER (5 CYCLES x 1/10")
0.01
0.1
1
10
100
1000
ω_{gc}
40
0
-40
-80
-50
-100
-150
-200
-250

Put $s = j\omega$

$$G(j\omega) = \frac{10}{j\omega(1+j\omega)}$$

Corner frequency $|1+j\omega| = 0$

$|j\omega| = 1$ & $\omega = 1$

Prepare a table

Term	CF	change in slope	New slope
$\frac{10}{j\omega}$	–	-20	–
$\frac{1}{1+j\omega}$	$\omega = 1$	-20	-40

$\omega = 0.1$,

gain at $\omega = 0.1$,

$= 20\log\frac{10}{\omega} = 20\log\frac{10}{0.1} = 40$

$\omega = 1$

gain at $\omega = 1$

$= 20\log\frac{10}{1} = 20$

$\omega = 10$

gain at $\omega = 10$

$= 20 - 40\log\left(\frac{10}{1}\right) = -20$

phase angle: $-90 - \tan^{-1}(\omega) =$

ω	phase angle
0.1	-95.7
0.5	-116.56
1	-135
5	-168.6
10	-174
50	-178.85
100	-179.42
500	-179.88

$\gamma = 21.6^\circ$

Step2:
Find $\phi_m = \gamma_n - \gamma_d + \epsilon$
$= 16 - 35 + 5$
$= -19 + 5 = -14^\circ$

Step3:
Find attenuation factor
$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = \frac{1 - \sin(-14)}{1 + \sin(-14)} = 0.99$

Step4: Find gain $-20\log\frac{1}{\sqrt{\alpha}} = -20\log\frac{1}{\sqrt{0.99}} = -0.04$

Step5: For this db gain find frequency from Bode plot
~~Also find~~ It is $\omega_m = 3.2$ rad/sec

Step 6: $T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{3.2\sqrt{0.99}}$

$= 0.314$

Step7: So find TF of compensator $= \cancel{\alpha}\frac{1}{\cancel{\alpha}}\frac{(1+sT)}{(1+s\alpha T)}$

$G_c(s) = \frac{1 + 0.314s}{1 + 0.31086s}$

Step8: The overall TF is
$G_c(s)G(s) = \frac{10}{s(s+1)}\frac{1+0.314s}{1+0.31086s}$
put $s = j\omega$ So $G_c(j\omega)G(j\omega) = \frac{10}{j\omega(1+j\omega)}\frac{1+0.314j\omega}{1+0.31086j\omega}$

Step9: Find expression for phase angle,
$= -90 - \tan^{-1}\omega + \tan^{-1}(0.314\omega) - \tan^{-1}(0.31086\omega)$
put $\omega = 2\omega_m$ we find phase angle $= -142^\circ$
So the phase margin $= 180 - 132 = 48^\circ$
Hence design is satisfied.

8. Using Routh criterion determine the stability of a system given by characteristic equation

$$s^4+8s^3+18s^2+16s+5=0$$

Comment on location of roots of characteristic equation? [Apr17]

s^4	1	18	5
s^3	8	16	0
s^2	16	5	0
s	13.5	0	0
s^0	5		

There is no sign change in first column of Routh array. So all roots lie on LHS of s plane

The system is stable

PART-B

1). a) Obtain the state model of the system described by the following transfer function $\frac{Y(s)}{U(s)} = \frac{5}{s^2+6s+7}$.

[M|J-2014]

Solution:-

$$\frac{Y(s)}{U(s)} = \frac{5}{s^2+6s+7}$$

$$Y(s)\,[s^2+6s+7] = 5U(s)$$

$$s^2Y(s) + 6sY(s) + 7Y(s) = 5U(s)$$

Taking inverse laplace transform,

$$\ddot{y} + 6\dot{y} + 7y = 5u$$

State variables,

$$x_1 = y,\ x_2 = \dot{y},\ \dot{x}_2 = \ddot{y}$$

$$\dot{x}_2 + 6x_2 + 7x_1 = 5u.$$

$$\boxed{\dot{x}_2 = 5u - 6x_2 - 7x_1}$$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\boxed{\dot{x}_1 = x_2}$$

State equation,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -7x_1 - 6x_2 + 5u.$$

State matrix,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u.$$

Output equation,

$$y = x_1$$

Output matrix,

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Obtain the transfer function model for the following state model system $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$. [M/J- 2014]

Solution:-

Transfer function, $\frac{Y(s)}{U(s)} = C[sI-A]^{-1}B + D$.

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & -1 \\ +6 & s+5 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\text{adj}\,[sI-A]}{|sI-A|}$$

$$|sI-A| = \begin{vmatrix} s & -1 \\ 6 & s+5 \end{vmatrix}$$

$$= s(s+5) + 6 = s^2 + 5s + 6$$

$$\text{adj}\,[sI-A] = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C[sI-A]^{-1}B+D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{s+5}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} \end{bmatrix}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s+5}{s^2+5s+6}}$$

2) Obtain the state transition matrix for the state model whose system matrix A is given by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. [M/J-2014]

Solution:-

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \phi(s) = L^{-1}[sI-A]^{-1}$$

$$sI-A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{|sI-A|} \operatorname{adj}[sI-A]$$

$$|sI-A| = \begin{vmatrix} s-1 & -1 \\ 0 & s-1 \end{vmatrix}$$

$$|sI-A| = (s-1)^2$$

$$\operatorname{adj}[sI-A] = \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s-1}{(s-1)^2} & \frac{1}{(s-1)^2} \\ 0 & \frac{s-1}{(s-1)^2} \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$$L^{-1}[sI-A]^{-1} = L^{-1} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}}$$

3) A system is characterised by the transfer function $\frac{Y(s)}{U(s)} = \frac{3}{s^3+5s^2+11s+6}$. Identify the first state as the output. Determine whether or not the system is completely controllable and observable.

[M|J-2013]

Solution:-

$$\frac{Y(s)}{U(s)} = \frac{3}{s^3+5s^2+11s+6}$$

$$Y(s)\left[s^3+5s^2+11s+6\right] = 3U(s)$$

$$s^3Y(s) + 5s^2Y(s) + 11sY(s) + 6Y(s) = 3U(s).$$

Taking inverse laplace transform,

$$\dddot{y} + 5\ddot{y} + 11\dot{y} + 6y = 3u.$$

State variable,

$$x_1 = y, \quad x_2 = \dot{y} = \dot{x}_1, \quad \dot{x}_2 = \ddot{y} = x_3, \quad \dot{x}_3 = \dddot{y}$$

$$\dot{x}_3 + 5x_3 + 11x_2 + 6x_1 = 3u$$

$\dot{x}_3 = 3u - 5x_3 - 11x_2 - 6x_1$
$\dot{x}_2 = x_3$
$\dot{x}_1 = x_2$

State matrix,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} u$$

Output equation,

$$y = x_1$$

Output matrix,

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Controllability:-

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -15 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -15 \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 42 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & -15 \\ 3 & -15 & 42 \end{bmatrix}$$

$$|Q_c| = 3(-9) = -27$$

$\boxed{|Q_c| \neq 0}$. Hence the system is controllable.

Observability:-

$$Q_o = \left[C^T \quad A^TC^T \quad (A^T)^2C^T\right]$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^TC^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T(A^TC^T) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|Q_o| = 1$$

$$\boxed{|Q_o| \neq 0.}$$

Hence the system is observable.

4. a) Check the controllability of the following state space system. [M/J- 2014]

$$\dot{x}_1 = x_2 + u_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -2x_2 - 3x_3 + u_1 + u_2.$$

Solution:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} [u_1 \; u_2]$$

Controllability,

$$Q_c = [B \quad AB \quad A^2B]$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -3 & -3 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \\ 7 & 7 \end{bmatrix}$$

$$Q_c = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 & -3 \\ 1 & 1 & -3 & -3 & 7 & 7 \end{array}\right]$$

$$|Q_c| = -1(-1) + [(-1)(7+9) + 1(7+9)]$$

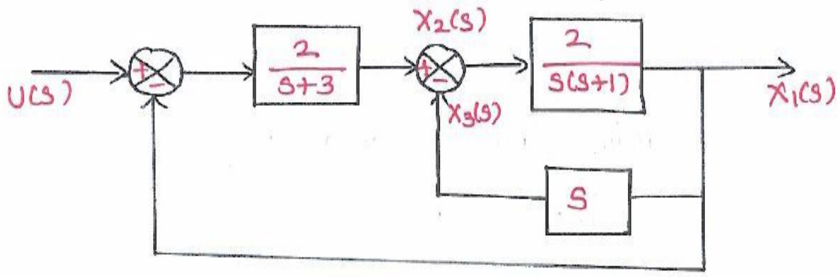
$$= 1 + [-1(16) + 1(16)]$$

$$|Q_c| = 1$$

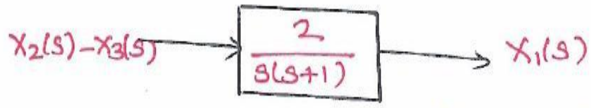
$$\boxed{|Q_c| \neq 0}$$

Hence the system is controllable.

b) Write the state equation for the system shown in figure. In which X_1, X_2, X_3 are the state vectors. Determine whether the system is completely controllable and observable. [N/D – 14, A/M – 11]



Solution:-



$$X_1(S) = \left[X_2(S) - X_3(S)\right]\left[\frac{2}{S(S+1)}\right]$$

$$S^2X_1(S) + SX_1(S) = 2X_2(S) - 2X_3(S)$$

Taking inverse laplace transform,

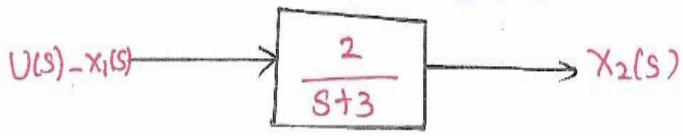
$$\boxed{\ddot{X}_1 + \dot{X}_1 = 2X_2 - 2X_3} \quad \rightarrow (1)$$

$$X_3(S) = SX_1(S)$$

Taking inverse laplace transform,

$$\boxed{X_3 = \dot{X}_1} \quad \rightarrow (2)$$

$$X_2(s) = [U(s) - X_1(s)]\left[\frac{2}{s+3}\right]$$



$$(s+3)X_2(s) = 2U(s) - 2X_1(s)$$

$$sX_2(s) + 3X_2(s) = 2U(s) - 2X_1(s)$$

Taking inverse laplace transform,

$$\dot{x}_2 + 3x_2 = 2u - 2x_1$$

$$\boxed{\dot{x}_2 = 2u - 2x_1 - 3x_2} \rightarrow (3)$$

$$\dot{x}_1 = x_3$$

$$\ddot{x}_1 = \dot{x}_3$$

From equation (1),

$$\dot{x}_3 + x_3 = 2x_2 - 2x_3$$

$$\dot{x}_3 = 2x_2 - 2x_3 - x_3$$

$$\boxed{\dot{x}_3 = 2x_2 - 3x_3}$$

State equation,

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = 2u - 2x_1 - 3x_2$$

$$\dot{x}_3 = 2x_2 - 3x_3$$

State matrix,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u.$$

Output equation,

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Controllability:-

$$Q_c = [B \quad AB \quad A^2B]$$

$$B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$$

$$|Q_c| = 4(8-0) = 32$$

$$\boxed{|Q_c| \neq 0}$$

Hence the system is controllable.

Observability:-

$$Q_0 = [C^T \quad A^TC^T \quad (A^T)^2C^T]$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^TC^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T(A^TC^T) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

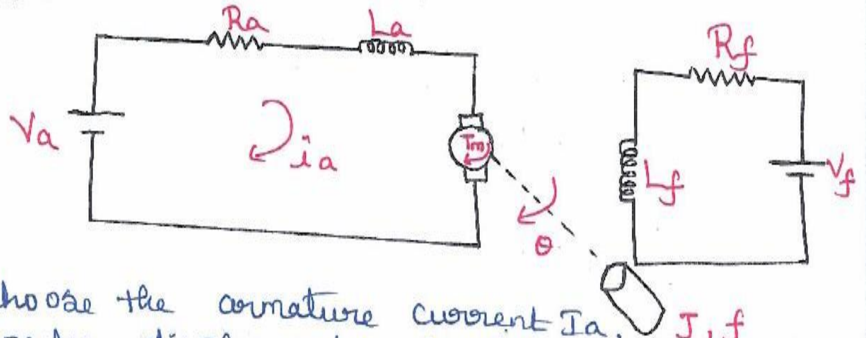
$$Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$|Q_0| = 1(-2) = -2$$

$$\boxed{|Q_0| \neq 0}$$

Hence the system is observable.

5) Obtain the state space representation of armature controlled DC motor with load shown below.

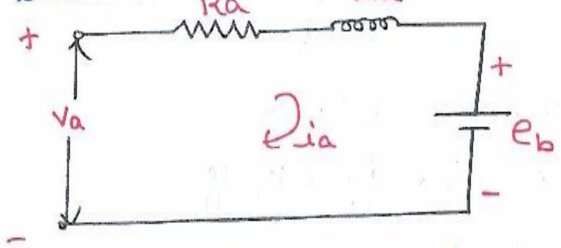


Choose the armature current I_a, the angular displacement of shaft θ, and the speed $d\theta/dt$ as state variables and θ as o/p variable.

[M/J-12]

The Speed of DC motor is directly proportional to armature voltage and inversely proportional to flux. In armature controlled DC motor, the desired Speed is obtained by Varying the armature Voltage.

The equivalent Circuit of armature is Shown below,



By Kirchoff's voltage law,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \rightarrow (1)$$

Where,

$i_a \rightarrow$ armature current.

$V_a \rightarrow$ armature voltage

$L_a \rightarrow$ armature inductance

$R_a \rightarrow$ armature resistance.

$e_b \rightarrow$ back emf.

Torque is directly proportional to the Product of flux and current. Since flux is constant, the torque is proportional to 'i_a' alone.

$$T \propto i_a$$

$$T = K_t i_a . \rightarrow (2)$$

where,

$K_t \rightarrow$ torque constant.

The differential equation governing the mechanical system are,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \rightarrow (3)$$

The back emf is proportional to speed.

$$e_b \propto \frac{d\theta}{dt}$$

$$e_b = K_b\frac{d\theta}{dt} \rightarrow (4)$$

Substitute eqn (4) in (1),

$$i_aR_a + L_a\frac{di_a}{dt} + K_b\frac{d\theta}{dt} = V_a. \rightarrow (5)$$

From eqn (2) & (3),

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_t i_a \rightarrow (6)$$

Eqn (5) & (6) are the differential equations governing the armature controlled DC motor.

Let us choose i_a, ω and θ as state variables to model the armature controlled DC motor.

$$x_1 = i_a, \; x_2 = \omega = \frac{d\theta}{dt} \text{ and } x_3 = \theta$$

The input to the motor is armature voltage V_a. Let $V_a = u$.

$$(5) \Rightarrow x_1 R_a + L_a\frac{dx_1}{dt} + K_b x_2 = u.$$

Let $\frac{dx_1}{dt} = \dot{x}_1$.

$$x_1 R_a + L_a \dot{x}_1 + K_b x_2 = u.$$

$$\boxed{\dot{x}_1 = -\frac{R_a}{L_a} x_1 - \frac{K_b}{L_a} x_2 + \frac{u}{L_a}} \rightarrow (7)$$

$$(6) \Rightarrow J \frac{d^2 x_3}{dt^2} + B \frac{dx_3}{dt} = K_t x_1$$

Let $\frac{d^2 x_3}{dt^2} = \dot{x}_2$ and $\frac{dx_3}{dt} = x_2$

$$J \dot{x}_2 + B x_2 = K_t x_1$$

$$\boxed{\dot{x}_2 = \frac{K_t}{J} x_1 - \frac{B}{J} x_2} \rightarrow (8)$$

The state variable $x_3 = \theta$.

$$\frac{dx_3}{dt} = \frac{d\theta}{dt}$$

Put $\frac{dx_3}{dt} = \dot{x}_3$ and $\frac{d\theta}{dt} = x_2$

$$\boxed{\dot{x}_3 = x_2.} \rightarrow (8)$$

State equation,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} & 0 \\ \frac{K_t}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} [u]$$

Let the desired outputs be i_a, ω and θ.

$$y_1 = i_a, \quad y_2 = \omega = \frac{d\theta}{dt} \text{ and } y_3 = \theta.$$

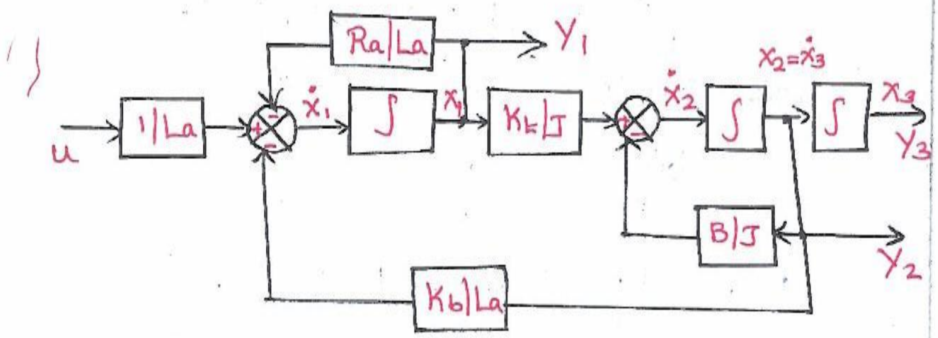
On relating the output variables to state variables we get,

$$y_1 = x_1, \quad y_2 = x_2 \text{ and } y_3 = x_3$$

The output equation in matrix form is,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The block diagram representation of the State model of armature controlled dc motor is shown below.



6. check the controllability & observability of the system whose state space representation is given as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ [Apr 2017]

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

By using Kalman's test $\quad Q_c = \left[B \,\vdots\, AB \,\vdots\, A^2B \right]$

$$B = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 21 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -10 \\ 8 \\ 21 \end{bmatrix} = \begin{bmatrix} 10 \\ -26 \\ -75 \end{bmatrix}$$

So $Q_c = \begin{bmatrix} 10 & -10 & 10 \\ 1 & 8 & -26 \\ 0 & 21 & -75 \end{bmatrix} = 10\{(8\times-75) - (-26\times21)\} - 1\{(-10\times-75) - (21\times10)\}$

$$= 10\{-600+546\} - 1\{750-210\}$$

$$= -540 - 540 = -1080 \neq 0$$

So System is controllable

To find observability

$$Q_o = \left[C^T \vdots A^T C^T \vdots (A^T)^2 C^T \right]$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

<u>$A^T C^T$:</u>

$$A^T = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T)$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{So } Q_o = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$Q_o = 0$ So it is not observable

7- obtain the time response of the system described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

with initial conditions $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

[NOV16] [Apr17]

Here $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

For such equations,

$$x(t) = L^{-1}[X(s)] = \int e^{A(t-\tau)} B u(\tau) d\tau + e^{At} x(0)$$

$$e^{(At)} = L^{-1}[sI - A]^{-1}$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{|sI - A|} \left[Adj [sI - A] \right]$$

$$= \frac{1}{s(s+2)+1} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{s^2+2s+1} & \frac{-1}{s^2+2s+1} \\ \frac{1}{s^2+2s+1} & \frac{s}{s^2+2s+1} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

I

$$\frac{s+2}{s^2+2s+1} = \frac{s+1}{(s+1)^2} + \frac{1}{(s+1)^2}$$

Taking inverse, $e^{-t} + te^{-t}$

$$\int_0^t (t-\tau)\, e^{(t-\tau)} = (t-\tau)\, e^{-(t-\tau)} + e^{-(t-\tau)}$$

$$\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau = \begin{bmatrix} e^{-(t-\tau)} + (t-\tau)e^{-(t-\tau)} + e^{-(t-\tau)} + (t-\tau)e^{-(t-\tau)} + e^{-(t-\tau)} \\ (t-\tau)e^{-(t-\tau)} + e^{-(t-\tau)} + e^{-(t-\tau)} + (t-\tau)e^{-(t-\tau)} + e^{-(t-\tau)} \end{bmatrix}_0^t$$

$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$= \begin{bmatrix} -te^{-t} \\ e^{-t} - te^{-t} \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 - te^{-t} \\ e^{-t} + 1 - te^{-t} \end{bmatrix}$$

8. Determine whether system described by following state model is completely controllable & observable? [NOV16]

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Kalman's test:

Controllability matrix $Q_C = \left[B \;\vdots\; AB \;\vdots\; A^2B \right]$

$$B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

So $Q_C = \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix} = 32 \neq 0$ Controllable.

Observability matrix is given by,

$$Q_O = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right]$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T (A^T C^T) = (A^T)^2 C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

So $Q_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} = -2 \neq 0$ So it is observable

9.(a) What are state variables? Explain state space formulation with its equations? [Apr17]

A set of variables which describe the system such that knowledge of these variables $t = t_0$ together with knowledge of inputs for $t \geq t_0$ completely determine the behaviour of system at any instant are called as state variables.

State of a system is given by state variables. For the system other variables are input & output variables.

State variables $= x_1, x_2, x_3 \cdots x_n$

input variables $= u_1, u_2, u_3 \cdots u_m$

output variables $= y_1, y_2, y_3 \cdots y_n$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m$$

So
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

→ State equation

Similarly the output equation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

So $\dot{X} = AX + BU$ State equation

$Y = CX + DU$ output equation

9(b) Given that $A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Compute state transition matrix

Here $A = A_1 + A_2$

$$e^{At} = e^{(A_1+A_2)t} = e^{A_1 t} e^{A_2 t}$$

$$e^{A_1 t} = L^{-1}[sI - A_1]^{-1}$$

$$sI - A_1 = \begin{bmatrix} s-\sigma & 0 \\ 0 & s-\sigma \end{bmatrix}$$

$$[sI - A_1]^{-1} = \frac{1}{(s-\sigma)^2}\begin{bmatrix} s-\sigma & 0 \\ 0 & s-\sigma \end{bmatrix} = \begin{bmatrix} \frac{1}{s-\sigma} & 0 \\ 0 & \frac{1}{s-\sigma} \end{bmatrix}$$

$$L^{-1}[sI - A_1]^{-1} = \begin{bmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{bmatrix} = e^{A_1 t}$$

$$e^{A_2 t} = L^{-1}[sI - A_2]^{-1}$$

$$sI - A_2 = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} = \begin{bmatrix} s & +\omega \\ -\omega & s \end{bmatrix}$$

$$[sI - A_2]^{-1} = \frac{1}{(s^2+\omega^2)}\begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{+\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

$$L^{-1}[sI - A^{*}]^{-1} = \begin{bmatrix} \cos\omega t & +\sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$e^{At} = e^{A_1 t} e^{A_2 t} = \begin{bmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$\begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}$$